

On Influence, Stable Behavior, and the Most Influential Individuals in Networks: A Game-Theoretic Approach

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Abstract

We propose *influence games*, a new class of graphical games, as a model of the behavior of large, but finite networked populations. Grounded in non-cooperative game theory, we introduce a new approach to the study of influence in networks that captures the strategic aspects of complex interactions in the network. We study computational problems on influence games, including the identification of the most influential nodes. We characterize the computational complexity of various problems in influence games, propose several heuristics for the hard cases, and design approximation algorithms, with provable guarantees, for the most influential nodes problem.

Keywords: Computational Game Theory, Social Network Analysis, Influence in Social Networks, Nash Equilibrium, Computational Complexity

1 Introduction

The influence of an entity on its peers is a commonly noted phenomenon in both online and real-life social networks. In fact, there is growing scientific evidence that suggests that influence can induce behavioral changes among the agents in a network. For example, recent work in medical social sciences posits the intriguing hypothesis that smoking (Christakis and Fowler, 2008), obesity (Christakis and Fowler, 2007), and even happiness (Fowler and Christakis, 2008) is contagious within a social network. These studies have been based on a real-world social network constructed from the data collected during the Framingham Heart Study, a decades old effort to look into the risk factors of cardiovascular diseases (Dawber, 1980). Regardless of the specific problem addressed, the underlying system under study in that research exhibits several core features. First, it is often very *large and complex*, with many individual entities exhibiting different behaviors and interactions. Second, the *network structure of complex interactions* is central. Third, the *directions and strengths of local influences* are highlighted as very relevant to the global behavior of the system as a whole.

The prevalence of systems and problems like the ones just described in the context of social medical science, combined with the obvious issue of often limited control over individuals, raises immediate, broad, difficult, and longstanding policy questions: e.g., *Can*

we achieve a desired objective, such as reducing the level of smoking, or controlling obesity via targeted, minimal interventions in a system? How do we optimally allocate our often limited resources to achieve the largest impact in such systems? Clearly, these issues are not exclusive to obesity, smoking, or happiness; similar issues arise in a large variety of settings: drug use, vaccination, crime networks, security, marketing, markets, the economy, and public policy-making and regulations, to name a few.

The work reported in this paper is in large part motivated by such questions and their broader implication.

Our Contribution

Our major contributions are: (1) a new approach to influence in networks grounded in non-cooperative game theory; (2) *influence games* as a new class of graphical games to model the behavior of individuals in networks; and (3) a theoretical and empirical study of computational aspects of influence games, including an algorithm for the identification of the most influential individuals.

2 Influence in Networks

A very important problem in social network analysis is the identification of the most “influential” individuals (see, e.g., (Wasserman and Faust, 1994; Kleinberg, 2007) and the references therein). We now provide a brief, informal description of our approach to the influence problem. Roughly speaking, in our approach we consider a set of individuals S in a network to be *most influential, with respect to some objective of interest, and preferences over all subsets of individuals*, if S is the *most preferred* subset among all those that satisfy the following condition: were the individuals in S to choose the behavior \mathbf{x}_S prescribed to them by some *stable outcome* of the system $\mathbf{x} \equiv (\mathbf{x}_S, \mathbf{x}_{-S})$ (which achieves the desirable objective of interest), then the *only* stable outcome of the system that remains consistent with their choices \mathbf{x}_S is \mathbf{x} itself. Said differently, once the most influential nodes follow the behavior \mathbf{x}_S prescribed to them by a stable outcome \mathbf{x} achieving the objective of interest, they become collectively so influential that their behavior forces every other individual to a unique choice of behavior! Our proposed concept of the most influential individuals is illustrated in Figure 1 with a very simple example.

Connection to Rational Calculus Models of Collective Action

The formal study of individual behavior in a collective setting originally began under the umbrella of “collective behavior” in sociology and social psychology. The classical treatment of collective behavior views individuals in a “crowd” as irrational beings with a lowered intellectual and reasoning ability. The proposition is that an increased level of suggestibility among the individuals facilitates the rapid spread of the homogeneous “mind of the crowd” (Le Bon, 1897; Park and Burgess, 1921; Blumer, 1939). Herbert Blumer’s work, in particular, popularized the classical theory of collective behavior well beyond academia and into such domains as police and the armed forces (McPhail, 1991, p. 9). However, this theory was subjected to much criticism primarily because it did not study empirical

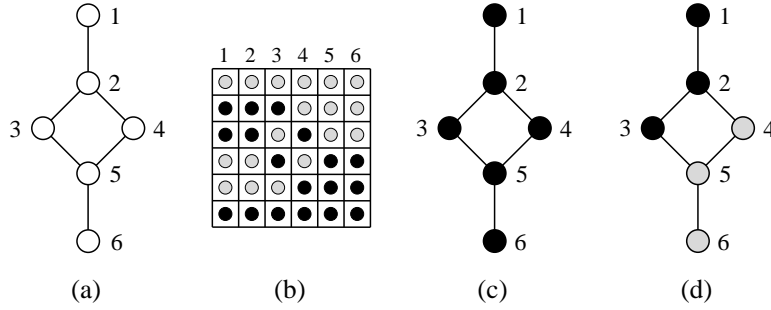


Figure 1: **Illustration of our approach to influence in networks.** Each node has a binary choice of behavior, $\{-1, +1\}$, and wants to behave like the majority of its neighbors (and is indifferent if there is a tie). We adopt pure-strategy Nash equilibrium (to be defined later), abbreviated as PSNE, as the notion of stable outcome. The network is shown in (a) and the enumeration of PSNE (a row for each PSNE, where black denotes node’s behavior 1, gray -1) in (b). We want to achieve the objective of every node choosing 1 (desirable outcome). Selecting the set of nodes $\{1, 2, 3\}$ and assigning these nodes behavior prescribed by the desirable outcome (i.e., 1 for each) lead to two consistent stable outcomes of the system, shown in (c) and (d). Thus, $\{1, 2, 3\}$ cannot be a most influential set of nodes. On the other hand, selecting $\{1, 6\}$ and assigning these nodes behavior 1 lead to the desirable outcome as the unique stable outcome remaining. Therefore, $\{1, 6\}$ is a most influential set, even though these two nodes are at the fringes of the network. Furthermore, note that $\{1, 6\}$ is not most influential in the diffusion setting, since it does not maximize the spread of behavior 1. (It should be mentioned that we study a much richer class of games in this paper than the one shown in this example.)

accounts systematically. Later on, Blumer himself referred to this as a “miserable job” by sociologists (Blumer, 1975).

In response to that, Clark McPhail undertook a massive effort, spanning three decades, to record the behavior of individuals in collective settings that he calls “gatherings” in order to distinguish it from (homogeneous) “crowds” in collective behavior (see McPhail (1991, ch. 5, 6) for a summary of his two-decade study). His empirical accounts, stored in a range of media formats as technology improved, reveal one common thing—that a gathering consists of individuals with diverse objectives, who nevertheless behave rationally and purposefully. To distinguish this purposive nature of individuals from irrationality in the classical treatment, he calls his study “collective action” and broadly defines it as “any activity that two or more individuals take with or in relation to one another” (McPhail, 2007, p. 881). In short, collective action can be seen as the modern approach, as opposed to the “old” (but not unimportant) approach of collective behavior (Miller, 2000, p. 14–15). A brief review of collective behavior and collective action literature is included in the Appendix for interested readers.

Many of the rational calculus or economic choice models that were originally proposed for collective behavior, are now discussed under collective action due to the purposive nature of the individuals. Here, we will conduct a very narrow and focused review of the relevant literature in order to place our model in its proper context. Our review will be concentrated around Mark Granovetter’s threshold models (Granovetter, 1978), which is one of the most

influential models of collective action to date.

Schelling's Models of Segregation

A notable precursor to Granovetter's threshold models is Nobel-laureate economist Thomas Schelling's models of segregation (Schelling, 1971, 1978). Schelling's models account for segregations that take place as a result of *discriminatory individual behavior* as opposed to organized processes (e.g., separation of on-campus residence between graduate and undergraduate students due to a university's housing policy) or economic reasons (e.g., segregation between the poor and the rich in many contexts). An example of a segregation due to individual choice, or "individually motivated segregation" as Schelling puts it (Schelling, 1971, p. 145), is the residential segregation by color in the U.S. Although Schelling's models and their analyses expressly focus on this case, these can be applied to many other scenarios as well.

In Schelling's *spatial proximity model*, if an individual's *level of tolerance* for population of the opposing type is exceeded in his neighborhood, he moves to another spatial location where he can be "happy." The dynamics of segregation is studied in this model using a rule of movement for the "unhappy" individuals. The *bounded-neighborhood model* is concerned with one global neighborhood. An individual enters it if it satisfies its level of tolerance constraint and leaves it otherwise. Schelling studies the stability of equilibria and the *tipping* phenomenon in this model when the distribution of tolerances and the population ratio of the two types are varied. An important finding is that in the cases studied, the modal level of tolerance does not correspond to a tipping point. More on Schelling's models can be found in the Appendix.

Berk's "Gaming" Approach

Another notable precursor to Granovetter's models is Berk's rational calculus approach (Berk, 1974). Berk strongly criticizes the assumption of individual irrationality which became prevalent in collective behavior literature. He formulates his approach by first giving a detailed empirical account of an anti-war protest at Northwestern University that originated in a town-hall meeting addressing dormitory rent hike. Berk's description gives accounts of both mundane and exciting happenings during the course of the protest and is recognized as "among the best in the literature" (McPhail, 1991, p. 126). He explains individual decision making through Raiffa's decision theory principles. To motivate his approach, he first notes that participating individuals were diverse in their disposition and that they exercised their reasoning power. He then broadly classifies the participants into two types—militants (with the desirable action of trashing properties) and moderates (with the desirable action of an anti-war activity, but not trashing). Each participant, militant or moderate, estimates the support in favor of his disposition, and with enough support, he will "act" (e.g., trash properties if he is militant). Clearly, an individual's estimate of support directly affects his "payoff." If an individual estimates that there is not enough support to act in favor of his disposition, he can try to persuade others to support his disposition so that he can receive a higher payoff by being able to act. This can be translated as an attempt to change others' payoff matrices, which is facilitated by the *milling* phase when they communicate and negotiate with each other. The milling phase ends when a

consensus or a compromise is reached and becomes common knowledge. A concerted action takes place at that time.

Granovetter's Threshold Models

Granovetter's threshold models (Granovetter, 1978) are presented in the setting of a crowd, where each individual is deciding whether to riot or not. In the simplest setting, each individual has a threshold and his decision is influenced by the decisions of others—if the number (or the proportion) of individuals already rioting is below his threshold, then he remains inactive, otherwise he engages in rioting. The emphasis is on investigating equilibrium outcomes due to the process of forward recursion (Granovetter, 1978, p. 1426), given a distribution of the thresholds of the population. It may be mentioned here that forward recursion starts only if there is an individual with a threshold of 0.

Granovetter's models are inspired by Schelling's models of segregation. In fact, one can draw a parallel between Schelling's level of tolerance and Granovetter's threshold in the following way. In Schelling's models, an individual leaves a neighborhood if his level of tolerance is exceeded, whereas in Granovetter's models, an individual becomes active in rioting if his threshold is exceeded. Furthermore, in both models, dynamics is of utmost importance and serves the purpose of explaining *how* an equilibrium collective outcome emerges from individual behavior. However, apart from these similarities, these two models are semantically different and also focus on completely different outlooks. First, Granovetter ascribes a deeper meaning to the concept of threshold. Threshold of an individual is not just “a number that he carries with him” from one situation to another (Granovetter, 1978, p. 1436). It rather depends on the situation in question and can even vary within the same situation due to changes occurring in it. Second, in Granovetter's models, a very small perturbation in the distribution of population threshold may lead to sharply different equilibrium outcomes. Granovetter highlights this property of his models as an explanation of seemingly paradoxical outcomes that goes against the predispositions of the individuals.

Two features of Granovetter's models make it stand out among the rational calculus models. First, the models are capable of capturing scenarios beyond the classical realm of collective behavior. Granovetter begins by setting up his model to complement the emergent norm theory (see Appendix) by providing an explicit model of how “individual preferences interact and aggregate” to form a new norm (Granovetter, 1978, p. 1421). Not only that such an explicit model eliminates the need for implicit assumptions (such as a new norm emerges when the majority of the population align themselves with that norm), it can also capture paradoxical outcomes alluded above that cannot be captured by the implicit assumption on the majority. Beyond the emergent norm theory, Granovetter's models can capture a wide range of phenomena that do not fall within the classical realm of collective behavior, such as diffusion of innovation, voting, public opinion, and residential segregation, to name a few. The second prominent feature of Granovetter's models is its ease of adaptation when dealing with a networked population. The same mechanism of forward recursion is applicable when the underlying influence structure is specified by a “sociomatrix,” which accounts for how much an individual influences another (Granovetter, 1978, p.1429). This is particularly useful for studying collective action in the setting of a social network.

Criticism of Rational Calculus Models

An implicit assumption regarding Granovetter’s sociomatrix is that the elements of the matrix are non-negative. Otherwise, the process of forward recursion may never terminate, even on the simplest of examples. However, many real world scenarios do exhibit co-existence of both positive and negative influences. For the most part, democrat senators in the U.S. Congress influence their republican colleagues negatively, while they influence colleagues of their own party positively. In residential segregation involving more than two types of individuals, an individual is negatively influenced, in different magnitudes, by individuals belonging to other types. Clearly, such a situation cannot be modeled using a non-negative sociomatrix. Furthermore, if we take a second look at Berk’s account, militant individuals positively reinforce each other in their decision to engage in trashing properties, whereas their decision is negatively affected by the moderates (that is, the presence of too many moderates makes it risky for militants to engage in violent action).

Critiques of rational calculus models point out the lack of behavioral adjustment in a “negative feedback” fashion (McPhail, 2007, p. 883). Here, negative feedback is defined in the context of the perceptual control theory that lays the foundation of McPhail’s *sociocybernetics theory* of collective action (see the Appendix). In a negative feedback system, an individual can adjust his behavior depending on the discrepancy between the input signal and the desired signal (the sign of this discrepancy has no correlation to negative feedback). In contrast, in a positive feedback system, such control of behavior is not possible. A typical example of a positive feedback system is a chemical chain reaction. An analogue to this is the “domino effect” cited often in rational calculus models (Granovetter, 1978, p. 1424). It is true that rational calculus models neither accounts for “errors” as desired by the proponents of the sociocybernetics theory, nor is it well-defined in the context of rational calculus. But it is not the case that “reversal” of behavior, which can be thought of as a crude form of behavioral adjustment, is precluded in rational calculus models. Such a form of behavioral adjustment can certainly be incorporated by allowing negative elements in the sociomatrix, but the challenge lies in the forward recursion process which may oscillate indefinitely because of those negative elements.

Our Approach

Although our approach may seem close to the rational calculus models of collective action, particularly to Granovetter’s threshold models, our objective is very much different from that of collective action theory. The focus of collective action theory in sociology is to explain *how* individual behavior in a group leads to collective outcomes. For example, Schelling’s models explain how different distributions of the level of tolerances of individuals lead to residential segregations of different properties. Berk explains how a compromise (such as placing a barricade) evolves within a mixture of rational individuals of different predispositions (militants vs. moderates). Granovetter shows how a little perturbation in the distribution of thresholds can possibly lead to a completely different collective outcome. In short, explaining collective social phenomena is at the heart of all these studies. While this is a scientific pursuit of utmost importance, our focus is rather on an engineering approach to *predicting* stable behavior in a networked population setting. Our approach is not to go through fine-grained details of a process, such as forward recursion, which is

often plagued with problems when the sociomatrix contains negative elements. Instead, we adopt the concept of Nash equilibrium to define stable outcomes. Said differently, the *path* to an equilibrium is not what we focus on; rather, it is the prediction of the equilibrium itself that we focus on. Next, we justify this approach.

Sociologists have recorded minute details of various collective action scenarios in order to substantiate their theories with empirical accounts. One example is Clark McPhail’s three-decade-old effort in recording a great many gatherings in various media formats. However, in the application scenarios that we are interested in, such as strategic interaction in the U.S. Congress and the U.S. Supreme Court, very little details can be obtained about *how* a collective outcome emerges. For example, the Budget Control Act of 2011 was passed by 74–26 votes in the Senate on August 2, 2011, ending a much debated debt-ceiling crisis. Despite intense media coverage, it would be difficult, if not impossible, to give an accurate account of how this agreement on debt-ceiling was reached. Even if there were an exact account of every conversation and every negotiation that had taken place, it would be extremely challenging to translate such a subjective account into a mathematically defined process, let alone learning the parameters and computing stable outcomes of such a complex model.

In addition, simplistic models of dynamics used in the literature require some restrictions on the underlying model. For example, as mentioned above, the forward recursion process implicitly assumes that the sociomatrix does not have negative elements. However, if we abstract the process, by the concept of Nash equilibrium in our case, we can deal with rich models without such a restriction and at the same time, capture equilibria beyond the ones captured by a simple model of dynamics. In particular, any equilibrium that the process of forward recursion converges to (with any initial configuration) is also captured by our model; but in addition, our model can capture equilibria that the forward recursion process cannot.

The basic intuition behind our approach is deeply rooted in the philosophy of AI and machine learning. For example, without fully understanding or modeling how human beings perform speech recognition, we have been able to device successful speech recognition systems. Not that the scientific question of how we perform speech recognition is not important, but the focus of AI, in general, is to engineer solutions that serve our purpose, not to explain physical phenomena. Interestingly enough, AI can sometimes help us understand physical phenomena, although not purposefully. Just as building aircrafts helped us understand the aerodynamics of a bird’s flight, recent research by psychologist Alison Gopnik suggests that young children, even 2-year-olds, perform Bayesian inference while learning from the environment (Gopnik et al., 2004).

In short, we propose an AI-based approach to predicting the behavior of a large, networked population. Our approach does not model the complex behavioral dynamics that takes place in the network, but abstracts it with the solution concept of Nash equilibrium. By doing this, we are able to deal with a rich set of models and focus more on the prediction of stable outcomes.

2.1 Connection to Literature on the Most Influential Nodes

To date, the study of influence in a network, by both economists (Morris, 2000; Chwe, 1999) and computer scientists (Kleinberg, 2007; Even-Dar and Shapira, 2007), has been rooted in rational calculus models of behavior. Their approach to connecting individual behavior to collective outcome is mostly by adopting the process of forward recursion (Granovetter, 1978, p. 1426), which is often employed in studying diffusion of innovations (Granovetter and Soong, 1983, p. 168). As a result, the term “contagion” in these settings has a rational connotation contrary to early sociology literature on collective behavior, where “contagion” or “social contagion” alludes to irrational and often hysteric nature of the individuals in a crowd (Park and Burgess, 1921; Blumer, 1939). The computational question of identifying the most influential nodes in a network (Kleinberg, 2007), which was originally posed by Domingos and Richardson (Domingos and Richardson, 2001), has also been studied using forward recursion within the context of rational calculus models. In the setting of Kleinberg and coauthors’ “cascade” or “diffusion models,” each node behaves in one of these two ways—it either adopts a new behavior or does not, and initially, none of the nodes adopts the new behavior. Given a number k , their formulation of the most influential nodes problem asks us to select a set of k nodes such that the *spread* of the new behavior is maximized by the selected nodes being the initial adopters of the new behavior. (Note that in their setting, the set of initial adopters, some of whom may have thresholds greater than 0, are *externally* selected in order to set off the forward recursion process, whereas in Granovetter’s setting, the initial adopters must have a threshold of 0.)

The notion of “most influential nodes” considered in this paper is different, and is aimed at *supplementing* the traditional line of work with a new game-theoretic perspective. In addition to the overview of our approach mentioned in the last subsection, let us briefly mention a few contrasting points between Kleinberg and coauthors’ approach to identifying the most influential nodes and that of ours.

A subtle aspect of diffusion models is that each node in the network behaves as an independent agent. Any observed influence that a node’s neighbors impose on the node is the result of the same node’s “rational” or “natural” response to the neighbors’ behavior. Thus, in many cases, it would be desirable that the solution to the most influential nodes problem lead us to a stable outcome of the system, in which each node’s behavior is a best response to the neighbors’ behavior. However, if we select a set of nodes with the goal of maximizing the spread of the new behavior then it might very well happen that some of the selected nodes are “unhappy” being the initial adopters of the new behavior relative to their neighbors’ final behavior at the end of forward recursion. For example, a selected node’s best behavioral response could be *not* adopting the new behavior after all. Thus, is not it more natural to require that the desired final state of the system, such as the maximum spread of the new behavior, be stable, in which everyone is “happy” with their behavioral response?

In order to address the question of finding the most influential nodes, the forward recursion process has been modeled as a “monotonic” process in general. (Here, a monotonic process refers to the setting where once an agent adopts the new behavior, it cannot go back.) If we think of an application such as reducing the incidence of smoking or obesity, then a model that allows a “change of mind” based on the response of the immediate

neighborhood may make more sense. Thus, a notable contrast between the traditional treatment of the most influential nodes problem and that of ours is that we do not restrict the influence among the nodes of the network to non-negative numbers. In fact, in many applications, both positive and negative influence factors may exist in the same problem instance. Take the U.S. Congress as an example: senators belonging to the same party may have non-negative influence factors on each other (as usually perceived from voting instances on legislation issues), but one senator may (and often does) have a negative influence on another belonging to a different party. While generalized versions of threshold models that allow “reversals” have been derived in the social science literature, to the best of our knowledge, there is no substantive work on the most influential nodes problem in that context.

Finally, the traditional approach to the most influential nodes problem emphasizes modeling the complex dynamics of interactions among the nodes in order to give the final answer, that is, a set of the most influential nodes. In fact, our model is inspired by the same threshold models that are used by them. However, as we have mentioned earlier, our emphasis is not on the dynamics of interactions, but on the stable outcomes in a game-theoretic setting. By doing this, we are able to capture significant, basic, and core *strategic* aspects of complex interaction in networks that naturally appear in many real-world problems (e.g., identifying the most influential senators in the U.S. Congress). Of course, we recognize the importance of the dynamics of interactions on capturing and studying problems of influence at a finer level of detail. Yet, we believe that our approach can still capture significant aspects of the problem even at the coarser level of “steady-state” or stable outcome.

2.2 A Brief Note on Mechanism Design

On the surface, our approach may seem related to mechanism design in objective. Yet, our approach is conceptually very different. Here, in contrast to a mechanism-design approach to achieving desirable stable outcomes, we are not interested in changing, defining, or engineering a new system—*the system is what it is*. We are rather interested in altering the behavior within the same system so as to lead or “tip” it to a desirable stable outcome. To this end, we cannot or need not change the system. Instead, we just need to “convince” the right individuals to adopt the behaviors prescribed by the desirable stable outcome: the others would follow “voluntarily” the behavior prescribed by that outcome because no other stable outcome is possible. Note that making the selected individuals follow the prescribed behavior is facilitated by the fact that they will end up “happy:” none will have an incentive to behave any other way after all!

2.3 Related Work in Game Theory

Other researchers have used similar notions of “influential individuals” in specific contexts. Particularly close to ours are the works of (Heal and Kunreuther, 2003, 2006, 2007), (Kunreuther and Heal, 2003), (Kunreuther and Michel-Kerjan, 2007), and (Kearns and Ortiz, 2003).

Also, our interest is on identifying an “optimal” set of influential nodes for a variety of optimality criteria based on the particular context. For instance, we may prefer the

set of influential individuals of minimal size. Such a preference is similar to the concept of “minimal critical coalitions” in the works of (Heal and Kunreuther, 2003, 2006, 2007) and (Kunreuther and Michel-Kerjan, 2007).

3 Influence Games

Inspired by threshold models (Granovetter, 1978), we introduce *influence games* as a model of influence in large networked populations. Even though the model falls within the general class of graphical games (Kearns et al., 2001), a distinctive feature of influence games is a very compact, parametric representation. Our emphasis is on the problem of computing stable outcomes of systems of influence and identifying influential agents within the network relative to a particular objective.

3.1 General Game-Theoretic Model

Let us first formalize *influence games* as a general model of behavior. Let n be the number of individuals in the population. For simplicity, we restrict our attention to binary behavior, a common assumption in most of the work in this area. Thus, $x_i \in \{-1, 1\}$ denotes the *behavior* of individual i , where $x_i = 1$ indicates that i “adopts” a particular behavior and $x_i = -1$ indicates i “does not adopt” the behavior. Some examples of behavior of this kind are supporting a particular political measure, candidate or party; holding a particular view or belief; vaccinating against a particular disease; installing virus protection software (and keeping it up-to-date); acquiring fire/home insurance; becoming overweight; taking up smoking; becoming a criminal or participating in criminal activity; among many others.

Definition 3.1. Denote by $f_i : \{-1, 1\}^{n-1} \rightarrow \mathbb{R}$ the function that quantifies the “influence” of other individuals on i . In influence games, we define the payoff function $u_i : \{-1, 1\}^n \rightarrow \mathbb{R}$ quantifying the preferences of each player i as $u_i(x_i, \mathbf{x}_{-i}) \equiv x_i f_i(\mathbf{x}_{-i})$, where \mathbf{x}_{-i} denotes the vector of all joint-actions excluding that of i .

Given $\mathbf{x}_{-i} \in \{-1, 1\}^{n-1}$, the *best-response correspondence* $\mathcal{BR}_i^{\mathcal{G}} : \{-1, 1\}^{n-1} \rightarrow 2^{\{-1, 1\}}$ of a player i of an influence game \mathcal{G} is defined as follows.

$$\mathcal{BR}_i^{\mathcal{G}}(\mathbf{x}_{-i}) \equiv \arg \max_{x_i \in \{-1, 1\}} u_i(x_i, \mathbf{x}_{-i}).$$

Therefore, for all individuals i and any possible behavior $\mathbf{x}_{-i} \in \{-1, 1\}^{n-1}$ of the other individuals in the population, the *best-response behavior* x_i^* of individual i to the behavior \mathbf{x}_{-i}^* of others satisfies

$$\begin{aligned} f_i(x_{-i}^*) > 0 &\implies x_i^* = 1, \\ f_i(x_{-i}^*) < 0 &\implies x_i^* = -1, \text{ and} \\ f_i(x_{-i}^*) = 0 &\implies x_i^* \in \{-1, 1\}. \end{aligned}$$

Informally, “positive influences” lead an individual to adopt the behavior, while “negative influences” lead the individual to “reject” the behavior; the individual is indifferent if there is “no influence.” A *stable outcome* of the system, by which we formally mean a *pure-strategy Nash equilibrium (PSNE)* of the corresponding influence game \mathcal{G} , is a behavior

assignment $\mathbf{x}^* \in \{-1, 1\}^n$ that satisfies all those conditions: Each player i 's behavior x_i^* is a (simultaneous) best-response to the behavior \mathbf{x}_{-i}^* of the rest. Denote the set of PSNE of game \mathcal{G} by

$$\mathcal{NE}(\mathcal{G}) \equiv \{\mathbf{x}^* \in \{-1, 1\}^n \mid x_i^* \in \mathcal{BR}_i^{\mathcal{G}}(\mathbf{x}_{-i}^*) \text{ for all } i\}.$$

3.2 Most Influential Nodes: Problem Formulation

In formulating the most influential nodes problem in a network, we depart from the traditional model of diffusion and adopt influence games as the model of strategic behavior among the nodes in the network.

Definition 3.2. Let \mathcal{G} be an influence game, $g : \{-1, 1\}^n \times 2^{[n]} \rightarrow \mathbb{R}$ be the goal or objective function mapping a joint-action and a subset of the players in \mathcal{G} to a real number quantifying the general preferences over the space of joint-actions and players' subsets, and $h : 2^{[n]} \rightarrow \mathbb{R}$ be the set-preference function mapping a subset of the players to a real number quantifying the a priori preference over the space of players' subsets. Denote by $\mathcal{X}_g^*(S) \equiv \arg \max_{\mathbf{x} \in \mathcal{NE}(\mathcal{G})} g(\mathbf{x}, S)$ the optimal set of PSNE of \mathcal{G} , with respect to g and a fixed subset of players $S \subset [n]$. We say that a set of nodes/players $S^* \subset [n]$ in \mathcal{G} is most influential with respect to g and h , if

$$S^* \in \arg \max_{S \subset [n]} h(S), \text{ s.t., } |\{\mathbf{x} \in \mathcal{NE}(\mathcal{G}) \mid \mathbf{x}_S = \mathbf{x}_S^*, \mathbf{x}^* \in \mathcal{X}_g^*(S)\}| = 1.$$

As mentioned earlier, we can interpret the players in S^* to be collectively so influential that they are able to restrict every other player's choice of action to a unique one: the action prescribed by some desirable stable outcome \mathbf{x}^* .

An example of a goal function g that captures the objective of achieving a specific stable outcome $\mathbf{x}^* \in \mathcal{NE}(\mathcal{G})$ is $g(\mathbf{x}, S) \equiv \mathbb{1}[\mathbf{x} = \mathbf{x}^*]$. Another example that captures the objective of achieving a stable outcome with the largest number of individuals adopting the behavior is $g(\mathbf{x}, S) \equiv \sum_{i=1}^n \frac{x_i + 1}{2}$.

A common example of the set-preference function h that captures the preference for sets of small cardinality is to simply define h such that $h(S) > h(S')$ iff $|S| < |S'|$.

3.3 Linear Influence Games

A simple instantiation of the general influence game model just described is the case of linear influences.

Definition 3.3. In a linear influence game (LIG), the influence function of each individual i is defined as $f_i(\mathbf{x}_{-i}) \equiv \sum_{j \neq i} w_{ji} x_j - b_i$ where for any other individual j , $w_{ji} \in \mathbb{R}$ is a weight parameter quantifying the "influence factor" that j has on i , and $b_i \in \mathbb{R}$ is a threshold parameter for i 's level of "tolerance" for negative effects.

It follows from Definition 3.1 that although the influence function of an LIG is linear, its payoff function is quadratic. Furthermore, the following argument shows that an LIG is a special type of graphical game in *parametric* form. In general, the influence factors w_{ji} induce a directed graph, where nodes represent individuals, and therefore, we obtain a graphical game having a *linear* (in the number of edges) representation size, as opposed to

the *exponential* (in the maximum degree of a node) representation size of general graphical games in normal form (Kearns et al., 2001). In particular, there is a directed edge (or arc) from individual j to i iff $w_{ji} \neq 0$.

3.4 Connection to Polymatrix Games

Polymatrix games (Janovskaja, 1968) are defined to be n -player noncooperative games where a player's total payoff is the sum of the *partial payoffs* received from the other players. For any joint action \mathbf{x} , player i 's payoff is given by $M_i(x_i, \mathbf{x}_{-i}) \equiv \sum_{j \neq i} \alpha_{ji}(x_j, x_i)$, where $\alpha_{ji}(x_j, x_i)$ is the partial payoff that i receives from j when i plays x_i and j plays x_j . Note that this partial payoff is local in nature and is not affected by the choice of actions of the other nodes. We will consider polymatrix games with only binary actions $\{1, -1\}$ here.

The following property shows an equivalence between LIGs and 2-action polymatrix games. Thus, our computational study of LIGs directly carries over to 2-action polymatrix games.

Proposition 3.4. *LIGs are equivalent to 2-action polymatrix games, modulo the set of PSNE.*

Proof. Assume that the number of players $n > 1$; otherwise, the statement holds trivially. We first show that given any instance of an LIG, we can design a polymatrix game that has the same set of PSNE. In an LIG instance, player i 's payoff is given by

$$\begin{aligned} u_i(x_i, \mathbf{x}_{-i}) &= x_i \left(\sum_{j \neq i} w_{ji} x_j - b_i \right) \\ &= x_i \sum_{j \neq i} \left(w_{ji} x_j - \frac{b_i}{n-1} \right) \\ &= \sum_{j \neq i} \left(x_i w_{ji} x_j - \frac{x_i b_i}{n-1} \right). \end{aligned}$$

Thus, constructing a polymatrix game instance by defining $\alpha_{ji}(x_j, x_i) \equiv x_i w_{ji} x_j - \frac{x_i b_i}{n-1}$, we have the same set of PSNE in both instances.

Next, we show the reverse direction. Player i 's payoff in a 2-action polymatrix game is given by

$$\begin{aligned}
M_i(x_i, \mathbf{x}_{-i}) &= \sum_{j \neq i} \alpha_{ji}(x_j, x_i) \\
&= \sum_{j \neq i} (\mathbb{1}[x_i = 1] \alpha_{ji}(x_j, 1) + \mathbb{1}[x_i = -1] \alpha_{ji}(x_j, -1)) \\
&= \sum_{j \neq i} \left(\frac{1+x_i}{2} \alpha_{ji}(x_j, 1) + \frac{1-x_i}{2} \alpha_{ji}(x_j, -1) \right) \\
&= \frac{x_i}{2} \sum_{j \neq i} (\alpha_{ji}(x_j, 1) - \alpha_{ji}(x_j, -1)) + \frac{1}{2} \sum_{j \neq i} (\alpha_{ji}(x_j, 1) + \alpha_{ji}(x_j, -1)).
\end{aligned}$$

Note that the second term above does not have any effect on i 's choice of action. Thus, we can re-define the payoff of player i , without making any change to the set of PSNE of the original polymatrix game, as follows.

$$\begin{aligned}
M'_i(x_i, \mathbf{x}_{-i}) &= \frac{x_i}{2} \sum_{j \neq i} (\alpha_{ji}(x_j, 1) - \alpha_{ji}(x_j, -1)) \\
&= \frac{x_i}{2} \left(\sum_{j \neq i} (\mathbb{1}[x_j = 1] \alpha_{ji}(1, 1) + \mathbb{1}[x_j = -1] \alpha_{ji}(-1, 1)) - \right. \\
&\quad \left. \sum_{j \neq i} (\mathbb{1}[x_j = 1] \alpha_{ji}(1, -1) + \mathbb{1}[x_j = -1] \alpha_{ji}(-1, -1)) \right) \\
&= \frac{x_i}{2} \left(\sum_{j \neq i} \left(\frac{1+x_j}{2} \alpha_{ji}(1, 1) + \frac{1-x_j}{2} \alpha_{ji}(-1, 1) \right) - \right. \\
&\quad \left. \sum_{j \neq i} \left(\frac{1+x_j}{2} \alpha_{ji}(1, -1) + \frac{1-x_j}{2} \alpha_{ji}(-1, -1) \right) \right) \\
&= \frac{x_i}{4} \left(\sum_{j \neq i} x_j (\alpha_{ji}(1, 1) - \alpha_{ji}(-1, 1) - \alpha_{ji}(1, -1) + \alpha_{ji}(-1, -1)) \right. \\
&\quad \left. + \sum_{j \neq i} (\alpha_{ji}(1, 1) + \alpha_{ji}(-1, 1) - \alpha_{ji}(1, -1) - \alpha_{ji}(-1, -1)) \right).
\end{aligned}$$

Therefore, we can construct an LIG that has exactly the same set of PSNE as the polymatrix game, in the following way. For any player i , define $b_i \equiv -\sum_{j \neq i} \frac{1}{4}(\alpha_{ji}(1, 1) + \alpha_{ji}(-1, 1) - \alpha_{ji}(1, -1) - \alpha_{ji}(-1, -1))$, and for any player i and any other player j , define $w_{ji} \equiv \sum_{j \neq i} \frac{1}{4}(\alpha_{ji}(1, 1) - \alpha_{ji}(-1, 1) - \alpha_{ji}(1, -1) + \alpha_{ji}(-1, -1))$. \square

4 Equilibria Computation in Influence Games

We first study the problem of computing and counting PSNE in LIGs. We show that several special cases of LIGs present us with attractive computational advantages, while the general problem is intractable unless $P = NP$. We present heuristics to compute PSNE in general LIGs.

4.1 Nonnegative Influence Factors

When all the influence factors are non-negative, an LIG is supermodular (Milgrom and Roberts, 1990; Topkis, 1979). In particular, the game exhibits what is called *strategic complementarity* (Bulow et al., 1985). Hence, the best-response dynamics converges in at most n rounds. From this, we obtain the following result.

Proposition 4.1. *The problem of computing a PSNE is in P for LIGs on general graphs with only non-negative influence factors.*

This property implies certain monotonicity of the best-response correspondences. More specifically, for each player i , if any subset of the other players “increases his/her strategy” by adopting the new behavior, then player i ’s best-response cannot be to abandon adoption (i.e., move from 1 to -1). In other words, once a player adopts the new behavior, it has no incentive to go back. This monotonicity property also follows directly from the linear threshold model. Strategic complementarity implies other interesting characterizations of the structure of PSNE in LIGs and the behavior of best-response dynamics. For example, it is not hard to see that such games always have a PSNE: If we start with the complete assignment in which either everyone is playing 1, or everyone is playing -1 , parallel/synchronous best-response dynamics converges after at most n rounds (Milgrom and Roberts, 1990). If both best-response processes starting with all -1 ’s and all 1’s converge to the same PSNE, then the PSNE is unique. Otherwise, any other PSNE of the game must be “contained” between the two different PSNE. We can also view this from the perspective of constraint propagation with monotonic constraints (Russell and Norvig, 2003).

4.2 Special Influence Structures and Potential Games

Several special subclasses of LIGs are potential games (Monderer and Shapley, 1996). This connection guarantees the existence of PSNE in such games.

Proposition 4.2. *If the influence factors of an LIG \mathcal{G} are symmetric (i.e., $w_{ji} = w_{ij}$, for all i, j), then \mathcal{G} is a potential game.*

Proof. We show that the game has a cardinal potential function,

$$\Phi(\mathbf{x}) = \sum_{t=1}^n x_t \left(\sum_{i \neq t} \frac{x_i w_{it}}{2} - b_t \right). \quad (1)$$

Consider any player j . The difference in j 's payoff for $x_j = 1$ and $x_j = -1$ (assuming all other players play \mathbf{x}_{-j} in both cases) is

$$\begin{aligned}
& u_j(1, \mathbf{x}_{-j}) - u_j(-1, \mathbf{x}_{-j}) \\
&= 1 \times \left(\sum_{i \neq j} x_i w_{ij} - b_j \right) - (-1) \times \left(\sum_{i \neq j} x_i w_{ij} - b_j \right) \\
&= 2 \times \left(\sum_{i \neq j} x_i w_{ij} - b_j \right).
\end{aligned} \tag{2}$$

Next, the difference in the potential function when j plays 1 and -1 is

$$\begin{aligned}
& \Phi(1, \mathbf{x}_{-j}) - \Phi(-1, \mathbf{x}_{-j}) \\
&= 1 \times \left(\sum_{i \neq j} \frac{x_i w_{ij}}{2} - b_j \right) + \sum_{t \neq j} x_t \left(\sum_{i \neq t} \mathbb{1}[i \neq j] \frac{x_i w_{it}}{2} - b_t \right) \\
&+ \sum_{t \neq j} x_t \left(\sum_{i \neq t} \mathbb{1}[i = j] \frac{1 \times w_{it}}{2} - b_t \right) - \\
&(-1) \times \left(\sum_{i \neq j} \frac{x_i w_{ij}}{2} - b_j \right) - \sum_{t \neq j} x_t \left(\sum_{i \neq t} \mathbb{1}[i \neq j] \frac{x_i w_{it}}{2} - b_t \right) \\
&- \sum_{t \neq j} x_t \left(\sum_{i \neq t} \mathbb{1}[i = j] \frac{(-1) \times w_{it}}{2} - b_t \right) \\
&= 2 \times \left(\sum_{i \neq j} \frac{x_i w_{ij}}{2} - b_j \right) + 2 \times \left(\sum_{t \neq j} \frac{x_t w_{jt}}{2} \right) \\
&= 2 \times \left(\sum_{i \neq j} x_i w_{ij} - b_j \right).
\end{aligned} \tag{3}$$

The last line follows due to the symmetric weights (i.e., $w_{ij} = w_{ji}$). \square

If, in addition, the threshold $b_i = 0$ for all i , the game is a *party-affiliation game*, and computing a PSNE in such games is PLS-complete (Fabrikant et al., 2004).

The following result is on a large class of games that we call *indiscriminate* LIGs, where for every player i , the influence weight, $w_{ij} \equiv \delta_i \neq 0$, that i imposes on every other player j is the same. The interesting aspect of this result is that these LIGs are potential games despite being possibly *asymmetric* and exhibiting strategic substitutability (due to negative influence factors).

Proposition 4.3. *Let \mathcal{G} be an indiscriminate LIG in which all δ_i for all i , have the same sign, denoted by $\rho \in \{-1, +1\}$. Then \mathcal{G} is a potential game with the following potential function $\Phi(\mathbf{x}) = \rho \left[(\sum_{i=1}^n \delta_i x_i)^2 - 2 \sum_{i=1}^n b_i \delta_i x_i \right]$.*

Proof. It is sufficient to show that the sign of the difference in the individual utilities of any player due to changing her action unilaterally, is the same as the sign of the difference in the corresponding potential functions. For any player j , the first difference is

$$\begin{aligned} & 1 \times \left(\sum_{i \neq j} \delta_i x_i - b_j \right) - (-1) \times \left(\sum_{i \neq j} \delta_i x_i - b_j \right) \\ &= 2 \left(\sum_{i \neq j} \delta_i x_i - b_j \right). \end{aligned} \tag{4}$$

The potential function when j plays 1,

$$\begin{aligned} & \Phi(x_j = 1, \mathbf{x}_{-j}) \\ &= \rho \left[\left(\sum_{i \neq j} \delta_i x_i + \delta_j \times 1 \right)^2 - 2 \sum_{i \neq j} b_i \delta_i x_i - 2b_j \delta_j \times 1 \right] \\ &= \rho \left[\left(\sum_{i \neq j} \delta_i x_i \right)^2 + \delta_j^2 + 2 \left(\sum_{i \neq j} \delta_i x_i \right) \delta_j - 2 \sum_{i \neq j} b_i \delta_i x_i - 2b_j \delta_j \right]. \end{aligned}$$

The potential function when j plays -1 ,

$$\begin{aligned} & \Phi(x_j = -1, \mathbf{x}_{-j}) \\ &= \rho \left[\left(\sum_{i \neq j} \delta_i x_i + \delta_j \times (-1) \right)^2 - 2 \sum_{i \neq j} b_i \delta_i x_i - 2b_j \delta_j \times (-1) \right] \\ &= \rho \left[\left(\sum_{i \neq j} \delta_i x_i \right)^2 + \delta_j^2 - 2 \left(\sum_{i \neq j} \delta_i x_i \right) \delta_j - 2 \sum_{i \neq j} b_i \delta_i x_i + 2b_j \delta_j \right]. \end{aligned}$$

Thus, the difference in the potential functions,

$$\Phi(x_j = 1, \mathbf{x}_{-j}) - \Phi(x_j = -1, \mathbf{x}_{-j}) = 4\rho\delta_j \left(\sum_{i \neq j} \delta_i x_i - b_j \right). \tag{5}$$

Since $\rho\delta_j > 0$, the quantities given in (4) and (5) have the same sign. \square

4.3 Tree-Structured Influence Graphs

The following result follows from a careful, non-trivial modification of the **TreeNash** algorithm (Kearns et al., 2001). Note that the running time of the **TreeNash** algorithm is exponential in the degree of a node and thus also exponential in the representation size of an LIG! In contrast, our algorithm is linear in the maximum degree and thereby linear in the representation size of an LIG. The complete proof follows a proof sketch.

Theorem 4.4. *There exists an $O(nd)$ time algorithm to find a PSNE, or to decide that there exists none, in LIGs with tree structures, where d is the maximum degree of a node.*

Proof Sketch. We use similar notations as in (Kearns et al., 2001). The modification of the **TreeNash** involves efficiently (in $O(d)$ time, not $O(2^d)$) determining the existence of a witness vector and constructing one, if it exists, at each node during the downstream pass, in the following way.

Suppose that an internal node i receives tables $T_{ki}(x_k, x_i)$ from its parents k , and that i wants to send a table $T_{ij}(x_i, x_j)$ to its unique child j . If for some parent k of i , $T_{ki}(-1, x_i) = 0$ and $T_{ki}(1, x_i) = 0$, then i sends the following table entries to j : $T_{ij}(x_i, -1) = 0$ and $T_{ij}(x_i, 1) = 0$. Otherwise, we first partition i 's set of parents into two sets in $O(d)$ time: $Pa_1(i, x_i)$ consisting of the parents k of i that have a unique best response \hat{x}_k to i 's playing x_i and $Pa_2(i, x_i)$ consisting of the remaining parents of i . We show that $T_{ij}(x_i, x_j) = 1$ iff

$$x_i(x_j w_{ji} + \sum_{k \in Pa_1(i, x_i)} \hat{x}_k w_{ki} + \sum_{t \in Pa_2(i, x_i)} \underbrace{(2 \times \mathbb{1}[x_i w_{ti} > 0] - 1)}_{t's \text{ action in witness vector}} w_{ti}) \geq 0,$$

from which we get a witness vector, if it exists. \square

Following is the complete proof of Theorem 4.4.

Proof. We denote any node i 's action by $x_i \in \{-1, 1\}$, its threshold by b_i , and the influence of any node i on another node j by w_{ij} . Furthermore, the set of parents of a node i is denoted by $Pa(i)$. The two phases of the modified TreeNash algorithm are described below.

1. **Downstream phase.** In this phase each node sends a table to its unique child. We denote the table that node i sends to its child j as $T_{ij}(x_i, x_j)$, which is indexed by the actions of i and j , and define the set of conditional best-responses of a node i to a neighboring node j 's action x_j as $BR_i(j, x_j) \equiv \{x_i \mid T_{ij}(x_i, x_j) = 1\}$. If $|BR_i(j, x_j)| = 1$ then we will abuse this notation by letting $BR_i(j, x_j)$ be the unique best-response of i to j 's action x_j .

The downstream phase starts at the leaf nodes. Each leaf node l sends a table $T_{lk}(x_l, x_k)$ to its child k , where $T_{lk}(x_l, x_k) = 1$ if and only if x_l is a conditional best-response of l to k 's choice of action x_k . Suppose that an internal node i obtains tables $T_{ki}(x_k, x_i)$ from its parents $k \in Pa(i)$, and that i needs to send a table to its child j . Once i receives the tables from its parents, it first computes (in $O(d)$ time) the following three sets that partition the parents of i based on the size of their conditional best-response sets when i plays x_i .

$$Pa_r(i, x_i) \equiv \{k \text{ s.t. } k \in Pa(i) \text{ and } |BR_k(i, x_i)| = r\}, \text{ for } r = 0, 1, 2.$$

This is how i computes the table $T_{ij}(x_i, x_j)$ to be sent to j : $T_{ij}(x_i, x_j) = 1$ if and only if there exists a *witness vector* $(x_k)_{k \in Pa(i)}$ that satisfies the following two conditions:

Condition 1. $T_{ki}(x_k, x_i) = 1$ for all $k \in Pa(i)$.

Condition 2. The action x_i is a best-response of node i when every node $k \in Pa(i)$ plays x_k and j plays x_j .

There are two cases.

Case I: $Pa_0(i, x_i) \neq \emptyset$. In this case, there exists some parent k of i for which both $T_{ki}(-1, x_i) = 0$ and $T_{ki}(1, x_i) = 0$. Therefore, there exists no witness vector that satisfies Condition 1, and i sends the following table entries to j : $T_{ij}(x_i, x_j) = 0$, for $x_j = -1, 1$.

Case II: $Pa_0(i, x_i) = \emptyset$. In this case, we will show that there exists a witness vector for $T_{ij}(x_i, x_j) = 1$ satisfying Conditions 1 and 2 if and only if the following inequality holds (which can be verified in $O(d)$ time). Below, we will use the *sign* function σ : $\sigma(x) = 1$ if $x > 0$, and $\sigma(x) = -1$ otherwise.

$$x_i \left(w_{ji}x_j + \sum_{k \in Pa_1(i, x_i)} w_{ki}BR_k(i, x_i) + \sum_{k \in Pa_2(i, x_i)} w_{ki}\sigma(x_i w_{ki}) - b_i \right) \geq 0. \quad (6)$$

In fact, if Inequality (6) holds then we can construct a witness vector in the following way: If $k \in Pa_1(i, x_i)$, then let $x_k = BR_k(i, x_i)$, otherwise, let $x_k = \sigma(x_i w_{ki})$. Since each parent k of i is playing its conditional best-response x_k to i 's choice of action x_i , we obtain, $T_{ki}(x_k, x_i) = 1$ for all $k \in Pa(i)$. Furthermore, Inequality (6) says that i is playing its best-response x_i to each of its parent k playing x_k and its child j playing x_j .

To prove the reverse direction, we start with a witness vector $(x_k)_{k \in Pa(i)}$ such that Conditions 1 and 2 specified above hold. In particular, Condition 2 can be written as:

$$x_i \left(w_{ji}x_j + \sum_{k \in Pa(i)} w_{ki}x_k - b_i \right) \geq 0. \quad (7)$$

The following line of arguments shows that Inequality (6) holds.

$$\begin{aligned}
& x_i w_{ki} \sigma(x_i w_{ki}) \geq x_i w_{ki} x_k, \text{ for any } k \in Pa_2(i, x_i) \\
\Rightarrow & x_i \sum_{k \in Pa_2(i, x_i)} w_{ki} \sigma(x_i w_{ki}) \geq x_i \sum_{k \in Pa_2(i, x_i)} w_{ki} x_k \\
\Rightarrow & x_i \left(w_{ji} x_j + \sum_{k \in Pa_1(i, x_i)} w_{ki} BR_k(i, x_i) + \sum_{k \in Pa_2(i, x_i)} w_{ki} \sigma(x_i w_{ki}) - b_i \right) \\
& \geq x_i \left(w_{ji} x_j + \sum_{k \in Pa(i)} w_{ki} x_k - b_i \right) \\
\Rightarrow & x_i \left(w_{ji} x_j + \sum_{k \in Pa_1(i, x_i)} w_{ki} BR_k(i, x_i) + \sum_{k \in Pa_2(i, x_i)} w_{ki} \sigma(x_i w_{ki}) - b_i \right) \\
& \geq 0, \text{ using Inequality (7).}
\end{aligned}$$

In addition to computing the table T_{ij} , node i stores the following witness vector $(x_k)_{k \in Pa(i)}$ for each table entry $T_{ij}(x_i, x_j)$ that is 1: if $k \in Pa_1(i, x_i)$, then $x_k = BR_k(i, x_i)$, otherwise, $x_k = \sigma(x_i w_{ki})$. The downstream phase ends at the root node z , and z computes a unary table $T_z(x_z)$ such that $T_z(x_z) = 1$ if and only if there exists a witness vector $(x_k)_{k \in Pa(z)}$ such that $T_{kz}(x_k, x_z) = 1$ for all $k \in Pa(z)$ and x_z is a best-response of z to $(x_k)_{k \in Pa(z)}$.

The time complexity of the downstream phase is dominated by the computation of the table at each node, which is $O(d)$. We visit every node exactly once. So, the downstream phase is completed in $O(nd)$. Note that if there does not exist any PSNE in the game then all the table entries computed by some node will be 0.

2. **Upstream phase.** In the upstream phase, each node sends instructions to its parents about which actions to play, along with the action that the node itself is playing. The upstream phase begins at the root node z . For any table entry $T_z(x_z) = 1$, z decides to play x_z itself and instructs each of its parents to play the action in the witness vector associated with $T_z(x_z) = 1$. At an intermediate node i , suppose that it has been instructed to play x_i by its child j which itself is playing x_j . The node i looks up the witness vector $(x_k)_{k \in Pa(i)}$ associated with $T_{ij}(x_i, x_j) = 1$ and instructs its parents to play according to that witness vector. This process propagates upward, and when we reach all the leaf nodes, we obtain a PSNE for the game. Note that we can find a PSNE in this phase if and only if there exists one.

In the upstream phase, each node sends $O(d)$ instructions to its parents. Thus, the upstream phase takes $O(nd)$ time, and the whole algorithm takes $O(nd)$ time. \square

4.4 Hardness Results

Computing PSNE in a general graphical game is known to be computationally hard (Gottlob et al., 2005). However, that result does not imply intractability in our problem, nor do the

proofs seem easily adaptable to our case. LIGs are a special type of graphical game with quadratic payoffs, or in other words a graphical, *parametric* poly-matrix game (Janovskaja, 1968), and thus have a more succinct representation than general graphical games ($O(nd)$ in contrast to $O(n2^d)$, where d is the maximum degree of a node). Next, we show that various interesting computational questions on LIGs are intractable, unless $P = NP$.

The central hardness question on LIGs (and also on 2-action polymatrix games) is settled by 1(a) below. Related to the most influential nodes problem formulation, 1(b) states that given a subset of players, it is NP-complete to decide whether there exists a PSNE in which this subset of players adopts the new behavior. A similar statement is given in 1(c).

A prime feature of our formulation of the most influential nodes is the uniqueness of the desirable stable outcome when the set of the most influential nodes adopt their behavior according to the desirable stable outcome. Deciding whether a given set of players fulfills this criterion (in the special case of a desirable outcome where this set of players adopts the new behavior) is shown to be co-NP-complete in 2.

As we will see later, in order to compute a set of the most influential nodes, it suffices to be able to count the number of PSNE of an LIG (to be more specific, it suffices to count the number of PSNE extensions for a given partial assignment to the players' actions). This problem is shown to be #P-complete in 3. Note that the #P-completeness result for LIGs even with star structure is in contrast to the polynomial-time counterpart for general graphical games with tree graphs, for which not only deciding the existence of a PSNE is in P, but also counting PSNE on general graphical games with tree graphs is in P. This result can be better appreciated by considering the representation sizes of LIGs and tree-structured graphical games, which are linear and exponential in the maximum degree, respectively.

Below, we first summarize the hardness results with an outline of proof, followed by complete proofs of individual statements.

Theorem 4.5. 1. *It is NP-complete to decide the following questions in LIGs.*

- (a) *Does there exist a PSNE?*
 - (b) *Given a designated non-empty set of players, does there exist a PSNE consistent with those players playing 1?*
 - (c) *Given a number $k \geq 1$, does there exist a PSNE with at least k players playing 1?*
2. *Given an LIG and a designated non-empty set of players, it is co-NP-complete to decide if there exists a unique PSNE with those players playing 1.*
3. *It is #P-complete to count the number of PSNE, even for special classes of the underlying graph, such as a bipartite or a star graph.*

Proof Sketch. The complete proofs can be found in the Appendix. The proof of 1(a) reduces the 3-SAT problem to an LIG that consists of a player for each clause and each variable of the 3-SAT instance. The influence factors among these players are designed such that the LIG instance possesses a PSNE if and only if the 3-SAT instance has a satisfying

assignment. Since the underlying graph of the LIG instance is always bipartite, we obtain as a corollary that the NP-completeness of that existence problem holds even for LIGs on bipartite graphs.

The proofs of 1(b), 1(c), and 2 use reductions from the *monotone one-in-three SAT* problem. For 1(b), given a monotone one-in-three SAT instance I , we construct an LIG instance J having a player for each clause and each variable of I . Again, we design the influence factors in such a way that I is satisfiable if and only if J has a PSNE. The reduction for 1(c) builds upon that of 1(b) with specifically designed *extra players* and additional connectivity in the LIG instance. Again, the gadgets used in the proof of 1(c) are extended for the proof of 2.

The proof of 3 uses reductions from the 3-SAT and the #KNAPSACK problem. The reduction from the 3-SAT problem is the same as that used in 1(a), and proof of the #P-hardness of the bipartite case is by showing that the number of solutions to the 3-SAT instance is the same as the number of PSNE of the LIG instance. On the other hand, to prove the claim of #P-completeness of counting PSNEs of LIGs having star graphs, we give a reduction from the #KNAPSACK problem. Given a #KNAPSACK instance, we create an LIG instance with a star structure among the players and with specifically designed influence factors such that the number of PSNE of the LIG instance is the same as the number of solutions to the #KNAPSACK instance. \square

Complete Proofs of Hardness Results

To enhance the clarity of the proofs we have reduced existing NP-complete problems to LIGs with binary actions $\{0, 1\}$, instead of $\{-1, 1\}$. We next show, via a linear transformation, that any LIG with actions $\{0, 1\}$ can be reduced to an LIG with the same underlying graph, but with actions $\{-1, 1\}$.

Reduction from $\{0, 1\}$ -action LIG to $\{-1, 1\}$ -action LIG. Consider any $\{0, 1\}$ -action LIG instance I , where the influence factors and the thresholds are denoted by the symbols w and b , respectively (see Definition 3.3). We next construct a $\{-1, 1\}$ -action LIG instance J with the same players that are in I and with influence factors $w'_{ji} \equiv \frac{w_{ji}}{2}$ (for any i and any $j \neq i$), thresholds $b'_i \equiv b_i - \sum_{j \neq i} \frac{w_{ji}}{2}$ (for any i). We show that \mathbf{x} is a PSNE of I if and only if \mathbf{x}' is a PSNE of J , where $x'_i = 2x_i - 1$ for any i .

By definition, \mathbf{x} is a PSNE of I if and only if for any player i ,

$$\begin{aligned}
& x_i \left(\sum_{j \neq i} x_j w_{ji} - b_i \right) \geq (1 - x_i) \left(\sum_{j \neq i} x_j w_{ji} - b_i \right) \\
& \Leftrightarrow (2x_i - 1) \left(\sum_{j \neq i} x_j w_{ji} - b_i \right) \geq 0 \\
& \Leftrightarrow x'_i \left(\sum_{j \neq i} \frac{x'_j + 1}{2} w_{ji} - b_i \right) \geq 0 \\
& \Leftrightarrow x'_i \left(\sum_{j \neq i} x'_j \frac{w_{ji}}{2} - \left(b_i - \sum_{j \neq i} \frac{w_{ji}}{2} \right) \right) \geq 0 \\
& \Leftrightarrow x'_i \left(\sum_{j \neq i} x'_j w'_{ji} - b'_i \right) \geq 0,
\end{aligned}$$

which is the equivalent statement of \mathbf{x}' being a PSNE of J . □

Theorem 4.6. *It is NP-complete to decide if there exists a PSNE in an LIG.*

Proof. Since we can verify whether a joint action is a PSNE or not in polynomial time, the problem is in NP. We use a reduction from the 3-SAT problem to show that the problem is NP-hard.

Let I be an instance of the 3-SAT problem. Suppose that I has m clauses and n variables. For any variable i we define C_i to be the set of clauses in which i appears, and for any clause k we define V_k to be the set of variables appearing in clause k . For any clause k and any variable $i \in V_k$, let $l_{k,i}$ be 1 if i appears in k in non-negated form and 0 otherwise. We now build an LIG instance J from I . In this game every clause as well as every variable is a player. Each clause k has arcs to variables in V_k , and each variable i has arcs to clauses in C_i . The structure of the graph is illustrated in Figure 2. We next define the thresholds of the players and the influence factors on the arcs. For any clause k , let its threshold be $1 - \epsilon - \sum_{i \in V_k} (1 - l_{k,i})$. Here, ϵ is a constant, and $0 < \epsilon < 1$. For any variable i let its threshold be $\sum_{k \in C_i} (1 - 2l_{k,i})$. The weight on the arc from any clause k to any variable $i \in V_k$ is defined to be $1 - 2l_{k,i}$, and that from any variable i to any clause $k \in C_i$ is $2l_{k,i} - 1$. We denote the action of any clause k by $z_k \in \{0, 1\}$ and that of any variable i by $x_i \in \{0, 1\}$.

First, we prove that if there exists a satisfying truth assignment in I then there exists a PSNE in J . Consider any satisfying truth assignment S in I . Let the players in J choose their actions according to their truth values in S , that is, 1 for *true* and 0 for *false*. Clearly, every clause player is playing 1. Next, we show that every player in J is playing its best response under this choice of actions.

We now show that no clause has incentive to play 0, given that the other players do not change their actions. In the solution S to I , every clause has a literal that is *true*.

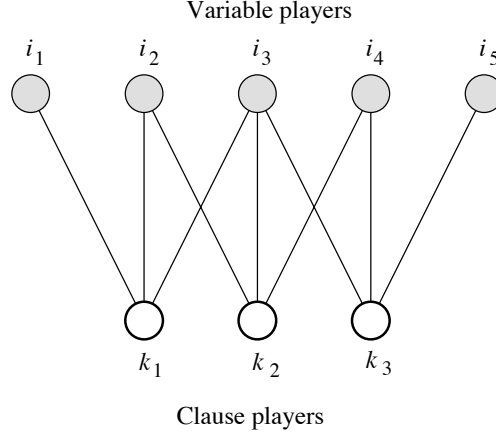


Figure 2: Illustration of the structure of an LIG instance from a 3-SAT instance (each undirected edge represents two arcs of opposite directions between the same two nodes). In this example, the 3-SAT instance is $(i_1 \vee i_2 \vee i_3) \wedge (\neg i_2 \vee i_3 \vee i_4) \wedge (\neg i_3 \vee i_4 \vee \neg i_5)$.

Therefore, in J every clause k has some variable $i \in V_k$ such that $x_i = l_{k,i}$. We have to show that the total influence on k is at least the threshold of k :

$$\begin{aligned}
& \sum_{i \in V_k} x_i (2l_{k,i} - 1) \geq 1 - \epsilon - \sum_{i \in V_k} (1 - l_{k,i}) \\
& \Leftrightarrow \sum_{i \in V_k} (x_i (2l_{k,i} - 1) + (1 - l_{k,i})) \geq 1 - \epsilon \\
& \Leftrightarrow \sum_{i \in V_k} (x_i l_{k,i} + (1 - x_i) (1 - l_{k,i})) \geq 1 - \epsilon.
\end{aligned}$$

Since for some $i \in V_k$, $x_i = l_{k,i}$, the above inequality holds strictly, that is,

$$\sum_{i \in V_k} (x_i l_{k,i} + (1 - x_i) (1 - l_{k,i})) > 1 - \epsilon.$$

Therefore, every clause k must play 1.

We need to show that no variable player has incentive to deviate, given that the other players do not change their actions. The total influence on any variable player i is $\sum_{k \in C_i} z_k (1 - 2l_{k,i}) = \sum_{k \in C_i} (1 - 2l_{k,i})$ (since $z_r = 1$ for every clause r). The threshold of i is $\sum_{k \in C_i} (1 - 2l_{k,i})$. Thus, every variable player i is indifferent between choosing actions 1 and 0 and has no incentive to deviate.

We now consider the reverse direction, that is, given a PSNE in J we show that there exists a satisfying assignment in I . We first show that at any PSNE, every clause must play 1. If this is not the case, suppose, for a contradiction, that for some clause r , $z_r = 0$.

Since r 's best response is 0 (this is a PSNE), we obtain

$$\begin{aligned} \sum_{i \in V_r} x_i(2l_{r,i} - 1) &\leq 1 - \epsilon - \sum_{i \in V_r} (1 - l_{r,i}) \\ \Leftrightarrow \sum_{i \in V_r} (x_i l_{r,i} + (1 - x_i)(1 - l_{r,i})) &\leq 1 - \epsilon. \end{aligned}$$

Therefore, for every variable player $j \in V_r$, $x_j \neq l_{r,j}$. Furthermore, for any $j \in V_r$, j does not have any incentive to deviate. Using these properties of a PSNE we will arrive at a contradiction, and thereby prove that z_r must be 1.

Consider any variable player $j \in V_r$, and let the difference between j 's total incoming influence and its threshold be U_j . We get

$$\begin{aligned} U_j &= \sum_{k \in C_j} z_k(1 - 2l_{k,j}) - \sum_{k \in C_j} (1 - 2l_{k,j}) = \sum_{k \in C_j} ((1 - z_k)(2l_{k,j} - 1)) \\ \Leftrightarrow U_j &= \sum_{k \in C_j} ((1 - z_k)(2l_{k,j} - 1)\mathbf{1}[l_{k,j} = 1]) + \sum_{k \in C_j} ((1 - z_k)(2l_{k,j} - 1)\mathbf{1}[l_{k,j} = 0]) \\ \Leftrightarrow U_j &= \sum_{k \in C_j} ((1 - z_k)\mathbf{1}[l_{k,j} = 1]) - \sum_{k \in C_j} ((1 - z_k)\mathbf{1}[l_{k,j} = 0]). \end{aligned}$$

At any PSNE, if $x_j = 1$ then $U_j \geq 0$; otherwise, $U_j \leq 0$. Thus, the best response condition for variable j gives us

$$\begin{aligned} \sum_{k \in C_j} ((1 - z_k)\mathbf{1}[l_{k,j} = x_j]) &\geq \sum_{k \in C_j} ((1 - z_k)\mathbf{1}[l_{k,j} \neq x_j]) \\ \Leftrightarrow \sum_{k \in C_j - \{r\}} ((1 - z_k)\mathbf{1}[l_{k,j} = x_j]) + (1 - z_r)\mathbf{1}[l_{r,j} = x_j] &\geq \\ \sum_{k \in C_j - \{r\}} ((1 - z_k)\mathbf{1}[l_{k,j} \neq x_j]) + (1 - z_r)\mathbf{1}[l_{r,j} \neq x_j] & \\ \Leftrightarrow \sum_{k \in C_j - \{r\}} ((1 - z_k)\mathbf{1}[l_{k,j} = x_j]) &\geq \\ \sum_{k \in C_j - \{r\}} ((1 - z_k)\mathbf{1}[l_{k,j} \neq x_j]) + 1, &\text{ since } l_{r,j} \neq x_j. \end{aligned}$$

The above inequality cannot be true, because the left hand side is always 0 (if $l_{k,j} = x_j$ then z_k must be 1 at any PSNE), and the right hand side is ≥ 1 . Thus, we have obtained a contradiction, and z_r cannot be 0.

So far, we have shown that at any PSNE $z_k = 1$ for any clause player k . To complete the proof, we now show that for every clause player k , there exists a variable player $i \in V_k$ such that $x_i = l_{k,i}$. If we can show this then we can translate the semantics of the actions in J to the truth values in I and thereby obtain a satisfying truth assignment for I .

Suppose, for the sake of a contradiction, that for some clause k and for all variable

$i \in V_k$, $x_i \neq l_{k,i}$. Since $z_k = 1$, we find that

$$\begin{aligned} \sum_{i \in V_k} x_i(2l_{k,i} - 1) &\geq 1 - \epsilon - \sum_{i \in V_k} (1 - l_{k,i}) \\ \Leftrightarrow \sum_{i \in V_k} (x_i l_{k,i} + (1 - x_i)(1 - l_{k,i})) &\geq 1 - \epsilon \\ \Leftrightarrow 0 &\geq 1 - \epsilon, \text{ which gives us the desired contradiction.} \end{aligned}$$

□

The proof of Theorem 4.6 reduces the 3-SAT problem to an LIG where the underlying graph is bipartite. Thus, we obtain the following corollary.

Corollary 4.7. *It is NP-complete to decide if there exists a PSNE in an LIG on a bipartite graph.*

The proof of Theorem 4.6 directly leads us to the following result that the counting version of the problem is #P-complete.

Corollary 4.8. *It is #P-complete to count the number of PSNE of an LIG.*

Proof. The proof follows from the proof of Theorem 4.6. Membership of this counting problem in #P is easy to see. Using the same reduction as in the proof of Theorem 4.6, we find that each satisfying truth assignment (among the 2^n possibilities) to the variables of the 3-SAT instance I can be mapped to a distinct PSNE of the LIG instance J . Furthermore, we have seen that at each PSNE in J , every clause player must play 1. Thus, for each of the 2^n joint strategies of the variable players (while having the clause players play 1), if the joint strategy is a PSNE then we can map it to a distinct satisfying assignment in I . Moreover, each of these two mappings are the inverse of the other. Therefore, the number of satisfying assignments of I is the same as the number of PSNE in J . Since counting the number of satisfying assignments of a 3-SAT instance is #P-complete, counting the number of PSNE of an LIG, even on a bipartite graph, is also #P-complete. □

While Corollary 4.8 shows the hardness of counting the number of PSNE of an LIG on a general graph, we can show the same hardness result even on special classes of graphs, such as star graphs:

Theorem 4.9. *Counting the number of PSNE of an LIG on a star graph is #P-complete.*

Proof. Since we can verify whether a joint strategy is a PSNE in polynomial time, the problem is in #P. We will show #P-hardness using a reduction from #KNAPSACK, which is the problem of counting the number of feasible solutions in a 0-1 Knapsack problem: Given n items, the weight $a_i \in \mathbb{Z}^+$ of each item i , and the maximum capacity of the sack $W \in \mathbb{Z}^+$, #KNAPSACK asks how many ways we can pick the items to satisfy $\sum_{i=1}^n a_i x_i \leq W$, where $x_i = 1$ if the i -th item has been picked, and $x_i = 0$ otherwise. Given an instance I of the #KNAPSACK problem with n items, we construct an LIG instance J on a star graph with $n + 1$ nodes. Let us label the nodes v_0, \dots, v_n , where v_0 is connected to all other nodes. We define the influence factors among the nodes as follows: the influence of v_0 to

any other node v_i , $w_{v_0 v_i} = 1$, and the influence in the reverse direction, $w_{v_i v_0} = -a_i$. The threshold of v_0 is defined as $b_{v_0} = -W$, and the threshold of every other node v_i , $b_{v_i} = 1$. We denote the action of any node v_i by $x_i \in \{0, 1\}$. Note that at any PSNE of J , v_0 must play 1. Otherwise, if v_0 plays 0 then all other nodes must also play 0, and this implies that v_0 must play 1, giving us a contradiction.

We prove that the number of feasible solutions in I is the same as the number of PSNE in J . For any $(x_1, \dots, x_n) \in \{0, 1\}^n$ in I , we map each x_i to the action selected by v_i in J , for $1 \leq i \leq n$. As proved earlier, the action of v_0 must be 1 at any PSNE. Furthermore, when v_0 plays 1, all other nodes become indifferent between playing 0 and 1. Thus, the number of PSNE in J is the number of ways of satisfying the inequality $\sum_{i=1}^n w_{v_i v_0} x_i \geq b_{v_0}$, which is equivalent to $\sum_{i=1}^n a_i x_i \leq W$. Thus the number of PSNE in J is equal to the number of feasible solutions in I . \square

The following three theorems show the hardness of several other variants of the problem of computing a PSNE of an LIG.

Theorem 4.10. *Given an LIG, along with a designated subset of k players in it, it is NP-complete to decide if there exists a PSNE consistent with those k players playing the action 1.*

Proof. It is easy to see that the problem is in NP, since a succinct yes certificate can be specified by a joint action of the players, where the designated players play 1, and it can be verified in polynomial time whether this is a PSNE or not.

We show a reduction from the monotone one-in-three SAT problem, a known NP-complete problem, to prove that the problem is NP-hard. An instance of the monotone one-in-three SAT problem consists of a set of m clauses and a set of n variables, where each clause has exactly three variables. The problem asks whether there exists a truth assignment to the variables such that each clause has exactly one variable with the truth value of *true*. Given an instance of the monotone one-in-three SAT problem, we construct an instance of LIG as follows (please refer to Figure 3 for an illustration). For each variable we have a *variable player* in the game, and for each clause we have a *clause player*. Each variable player has a threshold of 0, and each clause player has a threshold of ϵ , where $0 < \epsilon < 1$. We now define the connectivity among the players of the game. There is an arc with weight (or influence) -1 from a variable player u to another variable player v if and only if, in the monotone one-in-three SAT instance, both of the corresponding variables appear together in at least one clause. Also, for each clause t and each variable w appearing in t , there is an arc from the variable player (corresponding to w) to the clause player (corresponding to t) with weight 1. Furthermore, we assign $k = m$, and assume that the designated set of players is the set of clause players. We also assume that the action 1 in the LIG corresponds to the truth value of *true* in the monotone one-in-three SAT problem and 0 to *false*.

Note that the way we have constructed the LIG, at most one variable player per clause can play the action 1 at any PSNE. To see this, assume, for contradiction, that at some PSNE two variable players u and v , both connected to the same clause t , are playing the action 1. Then the influence on either of these two variable players is ≤ -1 , which is less than its threshold 0, and this contradicts the PSNE assumption. Also, note that at any

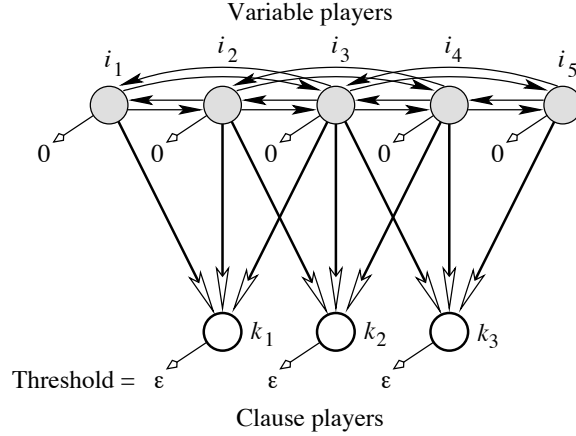


Figure 3: Illustration of the NP-hardness reduction of Theorem 4.10. The monotone one-in-three SAT instance is $(i_1 \vee i_2 \vee i_3) \wedge (i_2 \vee i_3 \vee i_4) \wedge (i_3 \vee i_4 \vee i_5)$. The threshold of each variable player is 0, and that of each clause player is ϵ .

PSNE, each clause player will play the action 1 if and only if at least one of the variable players connected to it plays 1.

First, we show that if there exists a solution to the monotone one-in-three SAT instance then there exists a PSNE in the LIG where the set of clause players play 1. A solution to the monotone one-in-three SAT problem implies that each clause has the truth value of *true* with exactly one of its variables having the truth value of *true*. We claim that in the LIG, every player playing according to its truth assignment, is a PSNE. First, observe that the variable players do not have any incentive to change their actions, since the ones playing 1 are indifferent between playing 0 and 1 (because the total influence = 0 = threshold) and the remaining must play 0 (because the total influence is $\leq -1 < \text{threshold}$). Since each clause has one of its variables playing 1, each clause player must play 1 (because $1 > \epsilon$). This concludes the first part of the proof.

We next show that if there exists a PSNE with the clause players playing 1 then there exists a solution to the monotone one-in-three SAT instance. Consider any PSNE where the clause players are playing 1. Since each clause player is playing 1, *at least* one of the three variable players connected to the clause player is playing 1. Furthermore, as we have shown earlier, no two variables belonging to the same clause can play 1 at any PSNE. Thus, for each clause player, *at most* one variable player connected to it is playing 1. Therefore, for every clause player, exactly one variable player connected to it is playing 1. Translating the semantics of the actions to the truth values of the variables and the clauses, we obtain a solution to the monotone one-in-three SAT instance. \square

Theorem 4.11. *Given an LIG and a number $k \geq 1$, it is NP-complete to decide if there exists a PSNE with at least k players playing the action 1.*

Proof. Clearly, the problem is in NP, since we can verify a whether a joint action is a PSNE or not in polynomial time.

For the proof of NP-hardness, once again we show a reduction from the monotone one-in-three SAT problem. Please see Figure 4 for an illustration. Given an instance I of the monotone one-in-three SAT problem, we first build an LIG as shown in the proof of Theorem 4.10. We then add $m(m-1)$ additional players, named *extra players*, to the game, where m is the number of clauses in I . Each of these extra players is assigned a threshold of ϵ , where $0 < \epsilon < 1$. The way we connect the extra players to the other players is as follows: From each clause player we introduce $m-1$ arcs, each weighted by 1, to $m-1$ distinct extra players. That is, no two clause players have arcs to the same extra player. Finally, we set $k = m^2$. We denote this instance of LIG by J .

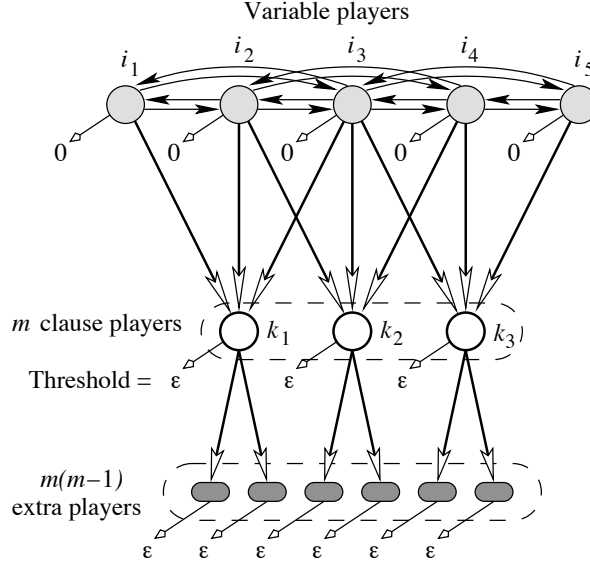


Figure 4: Illustration of the NP-hardness reduction of Theorem 4.11.

We prove that for any solution to I there exists a PSNE with k players playing 1 in J . Suppose that each of the variable and clause players is playing according to their corresponding truth value in the solution to I . None of the variable players has any incentive to change its action, because exactly one variable player connected to each clause player is playing 1. For the same reason, the clause players, each playing 1, also do not have any incentive to deviate. Considering the extra players, each of these players must play 1, because each of the clause players is playing 1. The total number of clause and extra players is k . Therefore, we have a Nash equilibrium where at least k players are playing 1.

On the other direction, consider any PSNE in J with at least k players playing 1. We claim that all the clause and extra players are playing 1 at this PSNE. If this is not true then at least one of these players is playing 0. This implies that at least one clause player is playing 0, because conditioned on a PSNE, whenever a clause player plays 1, all the extra players connected to it also plays 1. Furthermore, by our construction at most one of the variable players connected to each clause player can play 1. So, the total number of players playing 1 is $\leq (m-1)(m+1) < m^2$ (at most $m-1$ clause players are playing 1, and for each of these clause players, $m-1$ extra players, 1 variable player, and the clause player itself are playing 1), which contradicts our assumption that m^2 players are playing 1. Thus, at

any PSNE with k players playing 1, it must be the case that every clause player is playing 1. This leads us to a solution for I . \square

Theorem 4.12. *Given an LIG and a designated set of $k \geq 1$ players, it is co-NP-complete to decide if there exists a unique PSNE with those players playing the action 1.*

Proof. Two distinct joint actions (PSNE), each having the same k players playing 1, can serve as a succinct no certificate, and we can check in polynomial time if these two joint actions are indeed PSNE or not.

Suppose that I is an instance of the monotone one-in-three SAT problem. We reduce I to an instance J of our problem in polynomial time and show that J has a “no” answer if and only if I has a “yes” answer.

Given I , we start constructing an LIG in the same way as in Theorem 4.10 (see Figures 5 and 3). Assign $k = m^2$. Now, add two new players, named the *all-satisfied-verification player* and the *none-satisfied-verification player*, which have threshold values of $m - \epsilon$ and $-\epsilon$, respectively. We add arcs from every clause player to these two new players, and the arcs to the all-satisfied-verification player are weighted by 1, and the ones to the none-satisfied-verification player are weighted by -1 .

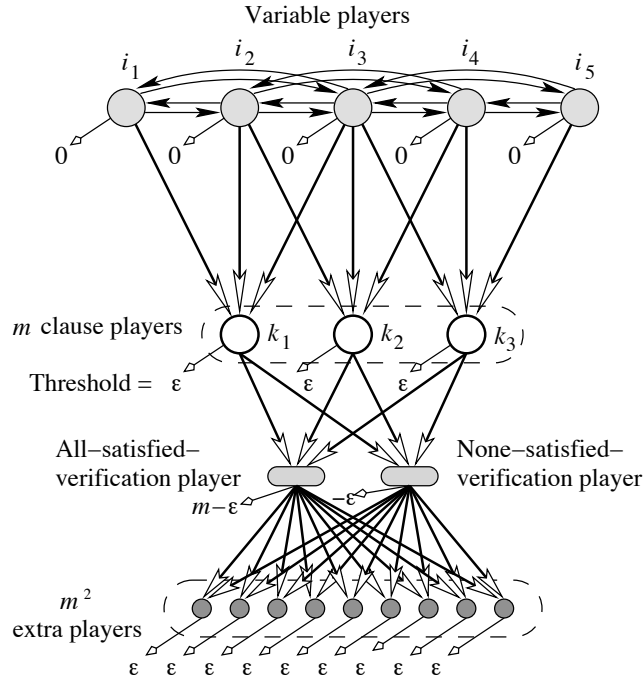


Figure 5: Illustration of the NP-hardness reduction (Theorem 4.12). For the monotone one-in-three SAT instance of Figure 3, we first obtain the same construction as in Theorem 4.10. We add two extra players, the all-satisfied-verification player and the none-satisfied-verification player, whose tasks are to verify if all clauses are satisfied and if no clause is satisfied, respectively. These two players are connected to m^2 extra players.

In addition, add $k = m^2$ new players, named *extra players*, and let these players constitute the set of designated players. Assign a threshold value of ϵ to each of these extra

players, and introduce new arcs, each with weight 1, from the all-satisfied-verification player and the none-satisfied-verification player to every extra player. The resulting LIG is the instance J of the problem in question.

Note that at any PSNE the all-satisfied-verification player plays 1 if and only if every clause player plays 1, and the none-satisfied-verification player plays 1 if and only if no clause player plays 1. Furthermore, at any PSNE, each extra player plays 1 if and only if either every clause player plays 1 or no clause player plays 1. Therefore, we find that every extra player playing 1, the none-satisfied-verification player playing 1, and every other player playing 0 is a PSNE, and we denote this equilibrium by E_0 . We claim that there exists a different PSNE where every extra player plays 1 if and only if I has a solution.

Suppose that there exists a solution S_I to I . It can be verified that making the all-satisfied-verification player play 1, none-satisfied-verification player play 0, every extra player play 1, and choosing the actions of the clause and the variable players according to the corresponding truth values in S_I gives us a PSNE that we call E_1 . Thus J has two PSNE E_0 and E_1 , where the k extra players play 1 in both cases.

Considering the reverse direction, suppose that there exists no solution to I . This implies that at any PSNE in J all clause players can never play 1, otherwise we could have translated the PSNE to a satisfying truth assignment for I . This further implies that the all-satisfied-verification player always plays 0. The none-satisfied-verification player plays 1 if and only if none of the clause players plays 1. Thus, every extra player plays 1 if and only if no clause player plays 1, if and only if no variable player plays 1. Therefore, E_0 is the only PSNE in J with the k extra players playing 1. \square

4.5 Heuristics for Computing and Counting Equilibria

The fundamental computational problem at hand is that of computing PSNE in LIGs. We have just seen that various computational questions pertaining to LIGs on general graphs, sometimes even on bipartite graphs, are NP-hard. We now present a heuristic to compute PSNE of an LIG on a general graph.

A natural approach to finding all the PSNE in an LIG would be to perform a backtracking search. However, a naive backtracking method that does not consider the structure of the graph would be destined to failure in practice. Thus, we need to order the node selections in a way that would facilitate pruning the search space.

The following is an outline of a backtracking search procedure that we have used in practice. The first node selected by the procedure is a node with the maximum outdegree. Intuitively, this node is the “most constraining” (see, e.g., Chapter 5 of (Russell and Norvig, 2003)) in terms of the number of nodes that a node directly influences. Subsequently, we select a node i that will most likely show that the current partial joint action cannot lead to a PSNE and explore the two actions of i , $x_i \in \{-1, 1\}$ in a suitable order. A good node selection heuristic that has worked well in our experiments is to select the one that has the maximum influence on any of the already selected nodes.

Suppose that the nodes are selected in the order $1, 2, \dots, n$ (wlog). After selecting node $i + 1$ and assigning it an action x_{i+1} , we determine if the partial joint action $\mathbf{x}_{1:(i+1)} \equiv (x_1, \dots, x_{i+1})$ can possibly lead to a PSNE and prune the corresponding search space if not. Note that a “no” answer to this requires a proof that one of the players j , $1 \leq j \leq i + 1$,

can never play x_j according to the partial joint action $\mathbf{x}_{1:(i+1)}$. A straightforward way of doing this is to consider each player j , $1 \leq j \leq i+1$, and compute the quantities $\gamma_j^+ \equiv \sum_{k=1, k \neq j}^{i+1} x_k w_{kj} + \sum_{k=i+2}^n |x_k w_{kj}|$ and $\gamma_j^- \equiv \sum_{k=1, k \neq j}^{i+1} x_k w_{kj} - \sum_{k=i+2}^n |x_k w_{kj}|$, and then test if the logical expression $((\gamma_j^- > b_j) \wedge x_j = -1) \vee ((\gamma_j^+ < b_j) \wedge x_j = 1)$ holds, in which case we can discard the partial joint action $\mathbf{x}_{1:(i+1)}$ and prune the corresponding search space. Furthermore, it may happen that due to $\mathbf{x}_{1:(i+1)}$, the choices of actions of some not-yet-selected players have become restricted. To this end, we apply **NashProp** (Ortiz and Kearns, 2002) with $\mathbf{x}_{1:(i+1)}$ as the starting configuration, and see if the choices of the other players have become restricted because of $\mathbf{x}_{1:(i+1)}$. Although each round of updating the table messages in **NashProp** takes exponential time in the maximum degree in general graphical games, we can show in a way similar to Theorem 4.4 that we can adapt the table updates to the case of LIGs so that it takes polynomial time.

A Divide-and-Conquer Approach

To further exploit the structure of the graph in computing the PSNE, we propose a divide-and-conquer approach that relies on the following separation property of LIGs.

Property 4.13. *Let $G = (V, E)$ be the underlying graph of an LIG and S be a vertex separator of G such that removing S from G results in $k \geq 2$ disconnected components: $G_1 = (V_1, E_1)$, ..., $G_k = (V_k, E_k)$. Let G'_i be the subgraph of G induced by $V_i \cup S$, for $1 \leq i \leq k$. Consider the LIGs on these (smaller) graphs G'_i 's, where we retain all the weights of the original graph, except that we treat the nodes in S to be indifferent (that is, we remove all the incoming arcs to these nodes and set their thresholds to 0). Computing the set of PSNE on G'_i 's and then merging the PSNE (by performing outer-joins of joint actions and testing for PSNE in the original LIG), we obtain the set of all PSNE of the original game.*

Proof Sketch. First, since the joint actions are tested for PSNE in the original LIG, the output will never contain a joint action that is not a PSNE. Second, since the nodes in S are made indifferent in the LIGs on G'_i , $1 \leq i \leq k$, no PSNE of the original LIG can get omitted from the result of the outer-join operation. \square

To obtain a vertex separator, we first find an edge separator (using well-known tools such as METIS (Karypis and Kumar, 1995)), and then convert the edge separator to a vertex separator (by computing a maximum matching on the bipartite graph spanned by the edge separator). We then use this vertex separator to compute PSNE of the game in the way outlined in Property 4.13. The benefits of this approach are two-fold: (1) for graphs that have good separation properties (such as preferential-attachment graphs), we have found this approach to be computationally effective in practice; and (2) this approach leads to an *anytime algorithm* for enumerating or counting PSNE: Observe that ignoring some edges from the edge separator may result in a smaller vertex separator, which greatly reduces the computation time of the divide-and-conquer algorithm at the expense of producing only a subset of all PSNE. (The reason we obtain a subset of all PSNE is that the edges that are ignored from the edge separator are not permanently removed from the original graph, and that after merging, every resulting joint action is tested for PSNE in the *original* game,

not in the game where some of the edges were temporarily removed. As a result, some the original PSNE may not be included in the final output. At the same time, we can never have a joint action in the final output that is not a PSNE.) We can obtain progressively better result as we ignore less number of edges from the edge separator.

5 Computing the Most Influential Nodes

We now focus on the problem of computing the most influential set of nodes with respect to a specified desirable PSNE and a preference for sets of minimal size. In the discussion below, we also assume, *only* for the purpose of establishing and describing the equivalence to the *minimum hitting set problem* (Karp, 1972), that we are given the set of all PSNE. (As we will see, a counting routine is all that our algorithm requires, not a complete list of PSNE.) We give a hypergraph representation of this problem that would lead us to a logarithmic-factor approximation by a natural greedy algorithm.

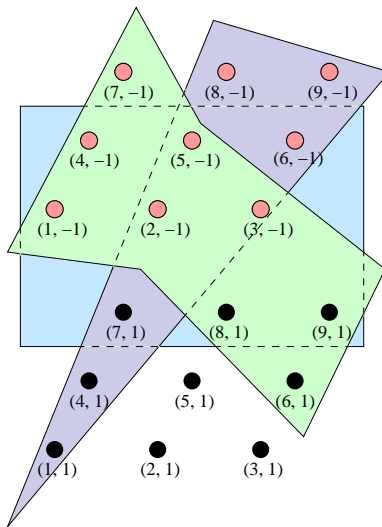


Figure 6: A hypergraph representation of three PSNE in a 9-player game with binary actions. The PSNE shown here are the followings: $(1, -1, -1, 1, -1, -1, 1, -1, -1)$ (triangle), $(-1, -1, -1, -1, -1, -1, 1, 1, 1)$ (rectangle), and $(-1, -1, -1, -1, -1, -1, 1, -1, 1)$ (6-gon).

Let us start by building a hypergraph that can represent the PSNE of a binary-action game. The nodes of this hypergraph are the player-action tuples of the game. Thus, for an n -player, binary-action game, we have $2n$ nodes in the hypergraph. That is, for each player i of the game, there are two nodes in the hypergraph: one in which i plays -1 (tuple $(i, -1)$, colored red in Figure 6) and the other in which i plays 1 (tuple $(i, 1)$, colored black). For every PSNE \mathbf{x} we construct a hyperedge $\{(i, x_i) \mid 1 \leq i \leq n\}$. Let us call this hypergraph the *game hypergraph*. By construction, a set of players S play the same joint-action $\mathbf{a}_S \in \{-1, 1\}^{|S|}$ in two distinct PSNE \mathbf{x} and \mathbf{y} of the LIG if and only if both of the corresponding hyperedges $e_{\mathbf{x}}$ and $e_{\mathbf{y}}$ (resp.) of the game hypergraph contains $T = \{(i, a_i) \mid i \in S\}$.

We can use the above property to translate the most influential nodes selection problem, given all PSNE, to an equivalent combinatorial problem on the corresponding game hypergraph H . Let $e_{\mathbf{x}^*}$ be the hyperedge in H corresponding to the desirable PSNE \mathbf{x}^* . Let us call $e_{\mathbf{x}^*}$ the *goal hyperedge*. Then the most influential nodes selection problem is the problem of selecting a minimum-cardinality set of nodes $T \subseteq e_{\mathbf{x}^*}$ such that T is contained in no other hyperedge of H (recall that we are dealing with a set-preference function that captures the preference for sets of minimal cardinality). Let us call the latter problem the *unique hyperedge problem*. Using the notation above, the equivalence relationship between the influential nodes selection problem (given the set of all PSNE) and the unique hyperedge problem can be stated as follows. The set $S \subseteq \{1, \dots, n\}$ is a (feasible) solution to the most influential nodes selection problem if and only if $T = \{(i, x_i^*) \mid i \in S\}$ is a (feasible) solution to the unique hyperedge problem.

We now show that the unique hyperedge problem is equivalent to the minimum hitting set problem. Immediate consequences of this result are that the unique hyperedge problem is not approximable within a factor of $c \log h$ for some constant $c > 0$, and that it admits a $(1 + \log h)$ -factor approximation (Raz and Safra, 1997; Johnson, 1974), where h is the total number of PSNE.

Theorem 5.1. *The unique hyperedge problem having $2n$ players and h hyperedges is equivalent to the minimum hitting set problem having n nodes and h hyperedges.*

Proof. Let us consider an instance I of the unique hyperedge problem, given by a game hypergraph $G = (V, E)$, where V is the set of $2n$ nodes and E is the set of h hyperedges, along with a specification of the goal hyperedge $e_{\mathbf{x}^*}$. Given I , we now construct an instance J of the minimum hitting set problem, specified by the hypergraph $G' = (e_{\mathbf{x}^*}, \{e_{\mathbf{x}^*}\} \cup \{\bar{e} \cap e_{\mathbf{x}^*} \mid e \in E \text{ and } e \neq e_{\mathbf{x}^*}\})$, where \bar{e} indicates the complement set of the hyperedge e . Thus, the nodes of G' are exactly the n nodes of $e_{\mathbf{x}^*}$ and the hyperedges of it are constructed from the complement hyperedges of G except $e_{\mathbf{x}^*}$, which is present in both G and G' . We show that a set S of nodes is a feasible solution to I if and only if it is a feasible solution to J .

If S is a feasible solution to I then $S \subseteq e_{\mathbf{x}^*}$ (because in the unique hyperedge problem, we are only allowed to select nodes from the goal hyperedge) and $S \not\subseteq e$ for any hyperedge $e \neq e_{\mathbf{x}^*}$ of G (otherwise, the uniqueness property is violated). This implies that for any hyperedge $e \neq e_{\mathbf{x}^*}$ of G , there exists a node $v \in S$ such that $v \notin e$, which further implies that $v \in \bar{e} \cap e_{\mathbf{x}^*}$. Thus, every hyperedge of G' , including $e_{\mathbf{x}^*}$, of course, has at least one of its nodes selected in S , and therefore, S is a feasible solution to J . On the other hand, if S is a feasible solution to J then for any hyperedge of G' , at least one of its nodes has been selected in S . That is, for any hyperedge $e \neq e_{\mathbf{x}^*}$ of G , we have $e' \equiv \bar{e} \cap e_{\mathbf{x}^*}$ as the corresponding complementary hyperedge in G' , and there exists a node $v \in S$ such that $v \in e'$, which implies that $v \notin e$. Thus, $S \not\subseteq e$ for any hyperedge $e \neq e_{\mathbf{x}^*}$ of G . Furthermore, all the nodes of S have been selected from $e_{\mathbf{x}^*}$ of G . Thus, $e_{\mathbf{x}^*}$ is the unique hyperedge of G containing the nodes of S .

To prove the reverse direction, we start with an instance J of the minimum hitting set problem, specified by a hypergraph $G' = (V, E)$, where V is a set of n nodes and E is a set of h hyperedges. Without the loss of generality, we assume that E contains the hyperedge e^* consisting of all the nodes of V . We now construct an instance I of the unique

hyperedge problem that has a hypergraph G with $2n$ nodes and h hyperedges. The node set of G literally consists of two copies of the nodes of V , denoted by $V \times \{1, -1\}$. We now construct the hyperedges of G . For each hyperedge $e \neq e^*$ of the minimum hitting set instance, we include a hyperedge $e' \equiv \bar{e} \times \{1\} \cup e \times \{-1\}$ in G , and for the hyperedge e^* of J , we include the hyperedge $e^* \times \{1\}$ in G . Thus, the game hypergraph can be defined as $G = (V \times \{1, -1\}, \{e^* \times \{1\}\} \cup \{\bar{e} \times \{1\} \cup e \times \{-1\} \mid e \in E \text{ and } e \neq e^*\})$. Finally, we designate $e^* \times \{1\}$ as the goal hyperedge of I . We will show that $S \subseteq V$ is a feasible solution to J if and only if $S \times \{1\}$ is a feasible solution to the unique hyperedge problem instance I . The set S is a feasible solution to J if and only if for every hyperedge $e \neq e^*$ of G' , there exists a node $v \in S$ such that $v \in e$ (note that $S \subseteq e^*$). This is equivalent to saying that for every hyperedge $e \times \{1\} \neq e^* \times \{1\}$ of G , there exists a node $v \in S \times \{1\}$ such that $v \notin e \times \{1\}$. Using the fact that $S \times \{1\} \subseteq e^* \times \{1\}$, $S \times \{1\}$ is a feasible solution to I . \square

The adaptation of the well-known hitting set approximation algorithm for our problem can be outlined as follows: At each step, select the least-degree node v of the goal hyperedge, remove the hyperedges that do not contain v , remove v from the game hypergraph, and include v in the solution set, until the goal hyperedge becomes the last remaining hyperedge in the hypergraph. In the context of the original LIG, at every round, this algorithm is essentially picking the node whose assignment would reduce the set of PSNE consistent with the current partial assignment the most. Hence, the algorithm only requires a subroutine to *count* the PSNE extensions for some given partial assignment to the players' actions, not an *a priori* full list or enumeration of all the PSNE. Of course, it may require a complete list of PSNE in the worst case.

6 Experimental Results

We have performed empirical studies on several types of LIGs, namely, random LIGs, preferential-attachment LIGs, LIGs created to model potential interactions in two different real-world scenarios: those among the U.S. Supreme Court Justices, and those among the U.S. senators. While the first two types of LIGs have been constructed artificially, the latter two have been learned from real-world data using machine learning techniques (Honorio and Ortiz, 2012).

6.1 Random Influence Games

As a first attempt, we have created instances of random graphs using the Erdős-Rényi model. The number of nodes have been varied from 10 to 30, and the probability of including an edge has also been varied. Assuming binary actions: 1 and -1 , the threshold b_i and the influence factors w_{ji} of the incoming arcs of each node i have been chosen uniformly at random from a unit hyperball. That is, for each node i , $b_i^2 + \sum_{j \in N(i)} w_{ji}^2 = 1$, where $N(i)$ is the set of nodes having arcs toward i . Then, the sign of each threshold, as well as each weight, has been chosen to be either $+$ or $-$ with a probability of 0.5. We have applied the heuristic given earlier to find the set of all PSNE in these random graphs. Our

experiments show that in all of these random LIGs, the number of PSNE, almost always, is very small—usually one or two, and sometimes none.

We have also studied LIGs on uniform random directed graphs. While constructing the random graphs, we have independently chosen each arc with a probability of 0.50, and assigned it a weight of -1 with a probability p (named *flip probability*) and 1 with probability $1 - p$. Several interesting findings have emerged from our study of this parameterized family of LIGs on uniform random graphs. The results are summarized in tabular forms in the Appendix. For various flip probabilities, we have independently generated 100 uniform random graphs of 25 nodes each, and for each of these random graphs, we have first computed all PSNE using our heuristic. We have then applied the greedy approximation algorithm to obtain a set of the most influential nodes in each graph and compared the approximation results to the optimal ones.

Unless p is either 0 or 1, the existence of a PSNE cannot be guaranteed. In our experiments, we have found that, in fact, for $p = 0.50$, the probability of not having a PSNE is highest (around 5%), and as we go toward the two extremes of p , the probability of not having a PSNE decreases. We have reported the games with at least one equilibrium in this experimental study, since these are the games that we are interested in for computing the most influential nodes. Another interesting finding with respect to the number of PSNE is that this number is very small when $p = 0$, that is when all the arcs have weight 1, and it is large when $p = 1$, although quite small (on average, a fraction $5.81 \times 10^{-6} \approx 2^{-17.29}$) relative to the total number of 2^{25} possible joint actions. Also, the average number of nodes of the search tree that the backtracking method visits per equilibrium computation is relatively small on the two extremes of p , compared with p around 0.5. Note that the backtracking method does a very good job with respect to the number of search-tree nodes visited in searching the 2^{25} space. In fact, our experiments have shown that the addition of the NashProp heuristic on top of the node selection heuristic considerably speeds up the search. Finally, we have found that although the approximation algorithm has a logarithmic factor worst-case bound, most often the results of the approximation algorithm are very close to the optimal solution.

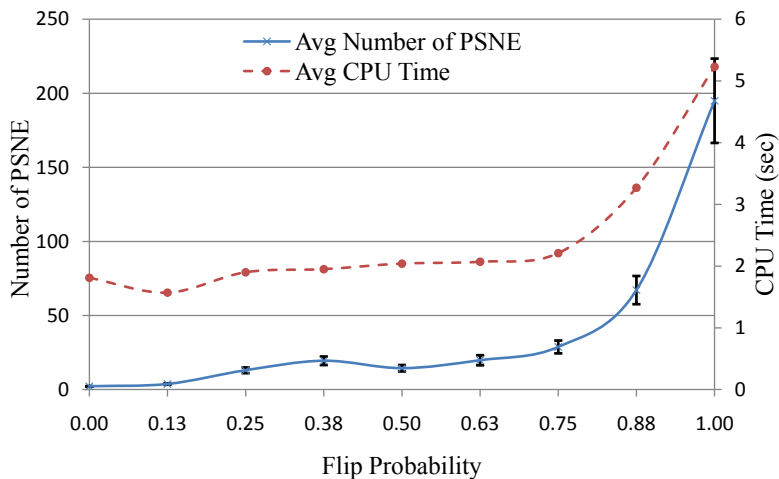


Figure 7: PSNE computation on random LIGs. The vertical bars denote 95% confidence intervals.

As shown in Figure 7, the number of PSNE usually increases if we have more negative-weighted arcs than positive ones, although the number of PSNE is still very small relative to the maximum potential number as remarked earlier. We have further found that although the approximation algorithm for influential nodes selection problem has a logarithmic factor worst-case bound, most often the result of the approximation algorithm is very close to the optimal solution. For example, for the random games having all negative influence factors, in 87% of the trials the approximate solution size \leq optimal size +1, and in 99% of the trials the approximate solution size \leq optimal size +2 (see the Appendix for more details in a tabular form).

6.2 Preferential-Attachment LIGs

We have also experimented with LIGs based on preferential-attachment graphs primarily because of its power to explain the structure of many real-world social networks in a generative fashion (Albert and Barabási, 2002). In order to construct these graphs, we have started with three nodes in a triangle and then progressively added each node to the graph, connecting it with three existing nodes with probabilities proportionate to the degrees. We have made each connection bidirectional and imposed the same weighting scheme as above: with the flip probability p , the weight of an arc is -1 and with probability $1 - p$ it is 1 . The threshold of each node has been set to 0 . We have observed that for $0 < p < 1$, these games have very few PSNE, while for $p = 0$ and $p = 1$ the number of PSNE is considerably larger than that. Furthermore, these games show very good separation properties, making the computation amenable to the divide-and-conquer approach. We show the average number of PSNE and the average computation time for graphs of sizes 20 to 50 nodes in Figure 8 for $p = 1$ (each average is over 20 trials). Note that in contrast to the random LIGs, preferential-attachment graphs show an exponential increase in the number of PSNE as the number of nodes increase, although the number of PSNE is still a very small fraction of the maximum potential number.

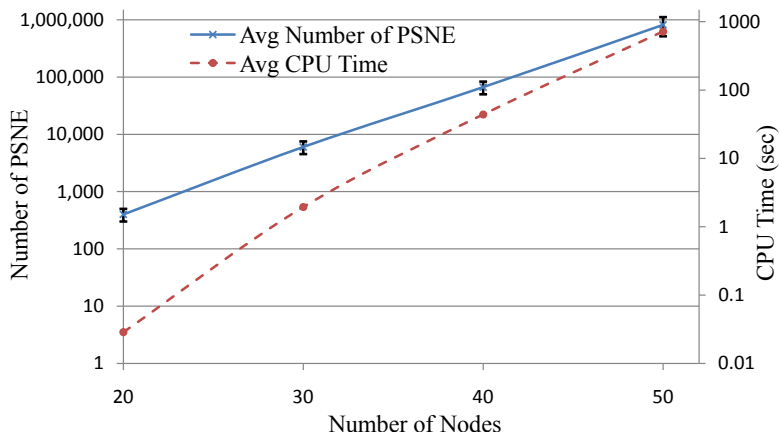


Figure 8: *PSNE computation on preferential-attachment LIGs (y-axis is in log scale). The vertical bars denote 95% confidence intervals.*

6.3 Illustration: Supreme Court Rulings

We have used our model to analyze the influence among the Justices of the U.S. Supreme Court. This is one of the application scenarios where the strategic aspects of influence is of prime importance. Two distinctive features of such a scenario are: first, the individual outcomes (in this case, the decisions of the Justices on each case) can be modeled as outcomes of a one-shot non-cooperative game (which in our case is LIG), and second, the physical interpretation of the diffusion process is not so much clear in such a scenario as it is in applications like viral marketing.

Data

We have obtained data from the Supreme Court Database.¹ Although the database captures fine-grained details of the cases, for our purpose we have only focused on the variable `varVote`. Again, the votes of the Justices are not simple yes/no instances. Instead, each vote can have eight distinct values. However, for practical purposes, we can attach a simple yes/no interpretation to the values of the votes, as shown in Table 1.

<code>varVote</code>	Original Meaning	Our Interpretation
1	Voted with majority	Yes
2	Dissent	No
3	Regular concurrence	Yes
4	Special concurrence	Yes
5	Judgment of the Court	Yes
6	Dissent from a denial or dismissal of certiorari, or dissent from summary affirmation of an appeal (Interpreted as absent from voting in final outcome)	Majority
7	Jurisdictional dissent (Interpreted as absent from voting in final outcome)	Majority
8	Justice participated in an equally divided vote	—

Table 1: Interpretation of Votes

In Table 1, “majority” in the third column signifies that we have interpreted the corresponding Justice’s vote as yes or no, whichever occurs most among the other Justices. Also, among the natural courts we studied, we did not encounter voting instances where `varVote` has a value of 8.

We next present our study of the natural court (with timeline 1994–2004) comprising of Justices WH Rehnquist, JP Stevens, SD O’Connor, A Scalia, AM Kennedy, DH Souter, C Thomas, RB Ginsburg, and SG Breyer.

¹<http://scdb.wustl.edu/>

Learning LIG

The data for the above natural court consists of 971 voting instances (each voting instance consists of the votes of all nine Justices). Many of these instances are repeated. For example, the most repeated instance is where all the Justices voted yes, which occurred 438 times. The second most repeated instance, which occurred 85 times, is where five of the Justices, namely, Justices Scalia, Thomas, Rehnquist, O’Connor, and Kennedy voted yes, while the others voted no. We have used L_2 -regularized logistic regression (simultaneous classification) to learn an LIG for this data. The influence factors and the biases of the LIG that has been learned is shown in a tabular form in the Appendix. A pictorial representation of the same LIG is shown in Figure 9.

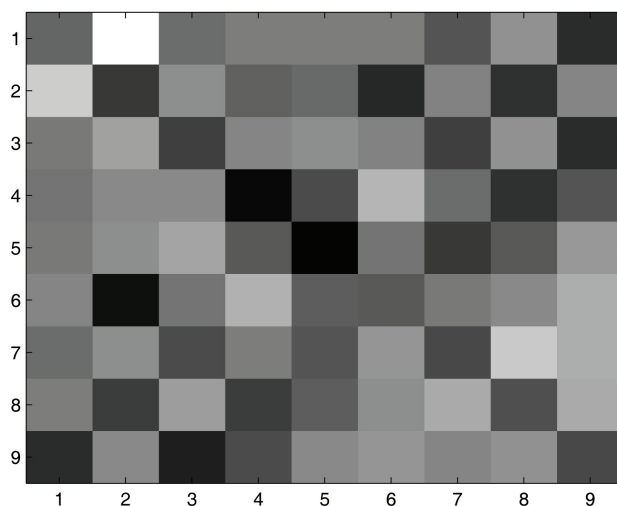


Figure 9: Pictorial representation of LIG learned from data—the non-diagonal elements represent influence factors and the diagonal elements biases. The numbering of the players (from 1 to 9) corresponds to the Justices in this order: Justices A Scalia, C Thomas, WH Rehnquist, SD O’Connor, AM Kennedy, SG Breyer, DH Souter, RB Ginsburg, and JP Stevens. The darker the color of a cell, the more negative is the corresponding number. For example, the most negative number (-0.2634) occurs in cell (5, 5) (i.e., the bias of Justice Kennedy). The most positive number (0.4282) occurs in cell (1, 2) (i.e., the influence factor from Justice Scalia to Justice Thomas) and the number closest to zero is 0.001 in cell (2, 4).

The learned LIG represents 589 of the 971 voting instances as PSNE. As expected, it represents the frequently repeated voting instances (such as the ones mentioned above). A graphical representation of the LIG is shown in Figure 10. We have clustered the nodes on the traditional perception that Justices Scalia, Thomas, Rehnquist, and O’Connor are “conservative;” Justices Breyer, Souter, Ginsburg, and Stevens are “liberal;” and Justice Kennedy is a “moderate.” As illustrated in Figure 10, negative influence factors occur only between players of two different clusters.

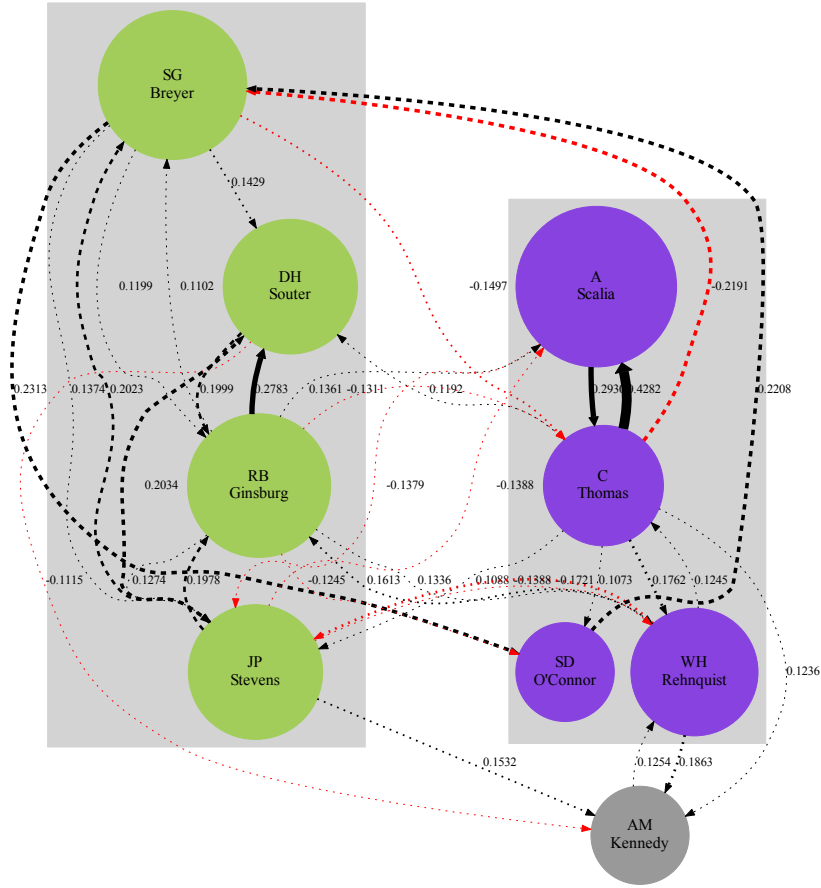


Figure 10: Graphical representation of LIG learned from data. Larger node sizes indicate higher thresholds (more stubborn). Positive influence factors are drawn as black arcs and negative as red. Thicker arcs represent higher value of influence factors. While the learned LIG is a complete graph, we have only drawn approximately half of the arcs (i.e., we are not showing the “weakest” arcs in this graph).

Most Influential Nodes

Analysis of the PSNE of this LIG shows that there is a set of two nodes that is most influential with respect to achieving the objective of every Justice voting yes. This most influential set consists of one node from the set {Scalia, Thomas} and another one from the set {Breyer, Souter, Ginsburg, Stevens}. Furthermore, any one node from the set {Breyer, Souter, Ginsburg, Stevens} is alone most influential with respect to achieving the objective of a 5-4 vote mentioned above (i.e., the second most repeated instance in the data).

6.4 Illustration: Congressional Voting

We further illustrate our computational scheme in another real-world scenario where the strategic aspects of the agents' behavior are of prime importance. We first learned the LIGs among the senators of the 101st and the 110th U.S. Congress (Honorio and Ortiz, 2012). The 101st Congress LIG consists of 100 nodes, each representing a senator, and 936 weighted arcs among these nodes. On the other hand, the 110th Congress LIG has the same number of nodes, but it is a little sparser than the 101st one, having 762 arcs. In these LIGs, each node can play one of the two actions: 1 (yes vote) and -1 (no vote). A bird's eye view of the 110th Congress LIG is shown in Figure 11 and a part of it is magnified and shown in Figure 12.

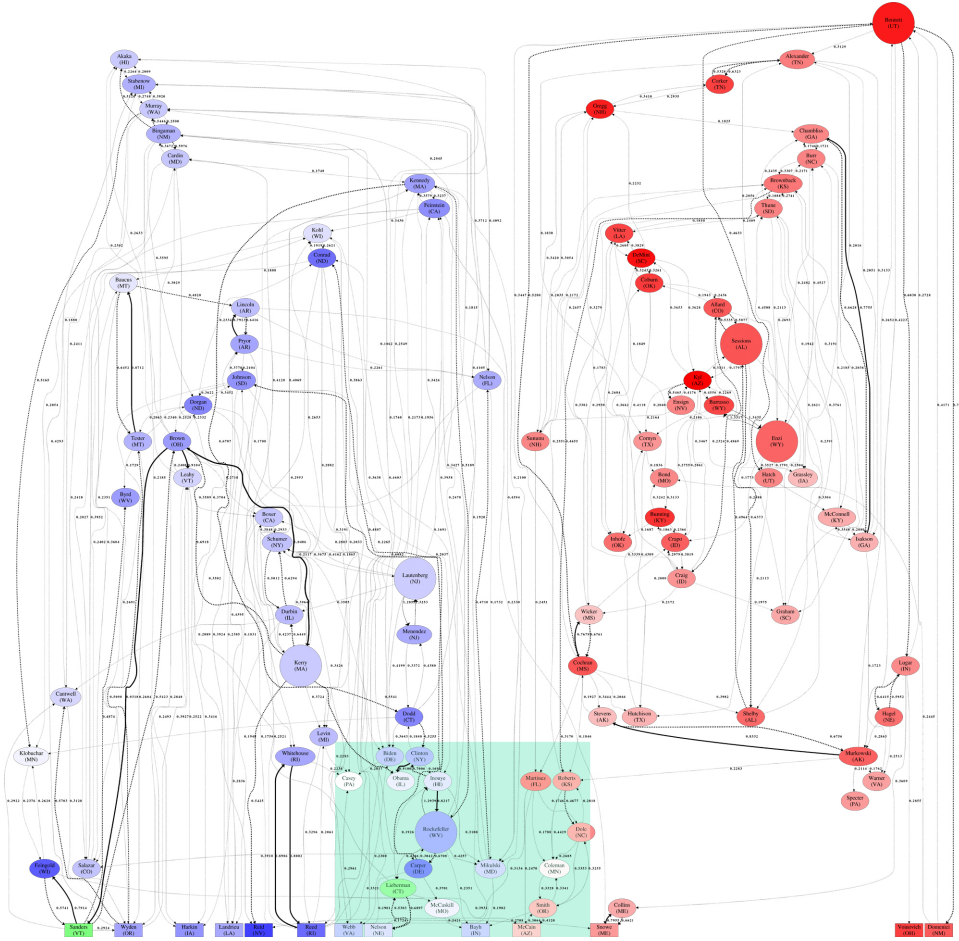


Figure 11: LIG for the 110th U.S. Congress: darker color of nodes represent higher threshold (more stubborn); thicker arcs denote influence factors of higher magnitude (only half of the original arcs with the highest magnitude of influence factors are shown here); circles denote most influential senators; rectangles denote cut nodes used in the divide-and-conquer algorithm. The shaded part of it has been magnified for better visualization in Figure 12.

First, we have applied the divide-and-conquer algorithm that exploits the nice separation

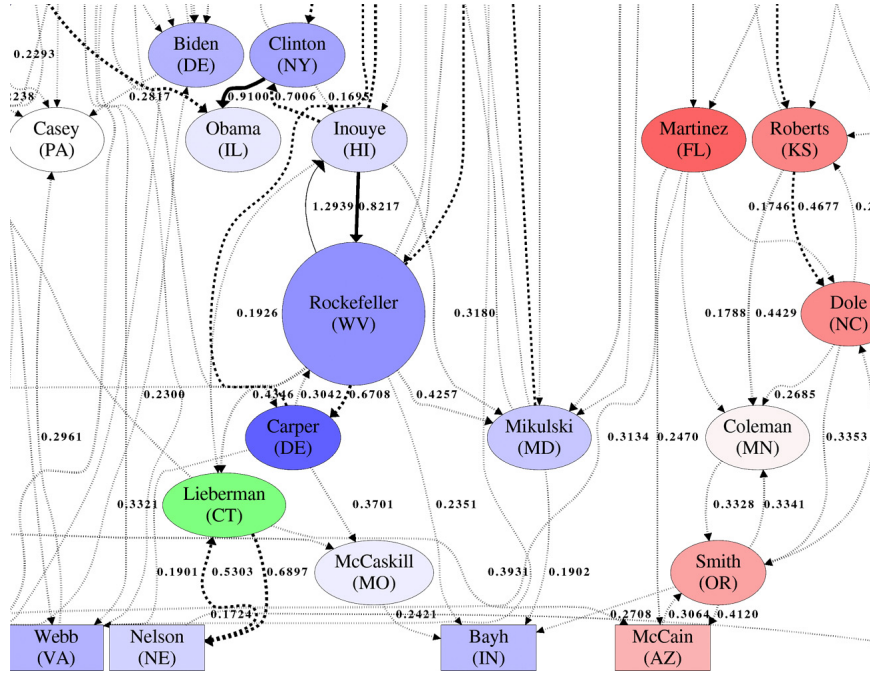


Figure 12: A part of the LIG for the 110th U.S. Congress: blue nodes represent Democrat senators, red Republican, and white independent; darker color of nodes represent higher threshold (more stubborn); thicker arcs denote influence factors of higher magnitude; circled node (Senator Rockefeller) denotes one of the most influential senators; rectangles at the bottom denote cut nodes used in the divide-and-conquer algorithm.

properties of these LIGs, to find the set of all PSNE (this has been done for convenience; as discussed earlier, counting alone would have been sufficient). We have obtained a total of 143,601 PSNE for the 101st Congress graph and 310,608 PSNE for the 110th one. Note that the number of PSNE in these games is extremely small (e.g., a fraction $2.45 \times 10^{-25} \approx 2^{-81.76}$ for the 110th Congress) relative to the maximum possible 2^{100} joint actions. Regarding the computation time, solving the 110th Congress using the divide-and-conquer approach takes about seven hours, whereas solving the same without this approach, simply relying on the backtracking search, takes about 15 hours on a modern quad-core desktop computer.

Next, we have computed the most influential senators using the approximation algorithm outlined earlier. We have obtained a solution of size five for the 101st Congress graph, which we have verified to be an optimal solution. This solution consists of Senators Rockefeller (Democrat, WV), Sarbanes (Democrat, MD), Thurmond (Republican, SC), Symms (Republican, ID), and Dole (Republican, KS). Interestingly, none of the maximum-degree nodes has been selected. Similarly, the six most influential senators of the more recent 110th Congress (January 2007–January 2009) are Kerry (Democrat, MA), Bennett (Republican, UT), Sessions (Republican, AL), Enzi (Republican, WY), Rockefeller (Democrat, WV), and Lautenberg (Democrat, NJ).

Contrasting with Diffusion Model

Our one-shot noncooperative game-theoretic model is fundamentally different from the diffusion model. However, comparing the most influential Senators (110th Congress) obtained using our setting to that obtained using diffusion revealed some striking similarity that we cannot yet explain fully. It should first be noted that both of these analyses have been done using the same influence factors and thresholds that we obtained from learning LIGs. In particular, for the diffusion setting, at each iteration we select a node u that achieves the maximum spread of action 1, force u to adopt action 1, let all but the previously selected nodes modify their actions as best responses to u 's adoption of action 1. We repeat this until every node adopts action 1.² *Note that because of negative influence factors, cycling may occur and this procedure may never come to a stop.* However, in our case, even in the presence of negative influence factors, we did not encounter such cycling. Furthermore, it is well known that the above recipe produces a provable approximation algorithm for the cascade model with submodular spread function (Kleinberg, 2007), but *this claim of approximation guarantee vanishes as soon as we have negative influence factors.*

We can visualize all possible choices of the most influential nodes that an algorithm can make as a directed acyclic graph, as shown in Figures 13 and 14.

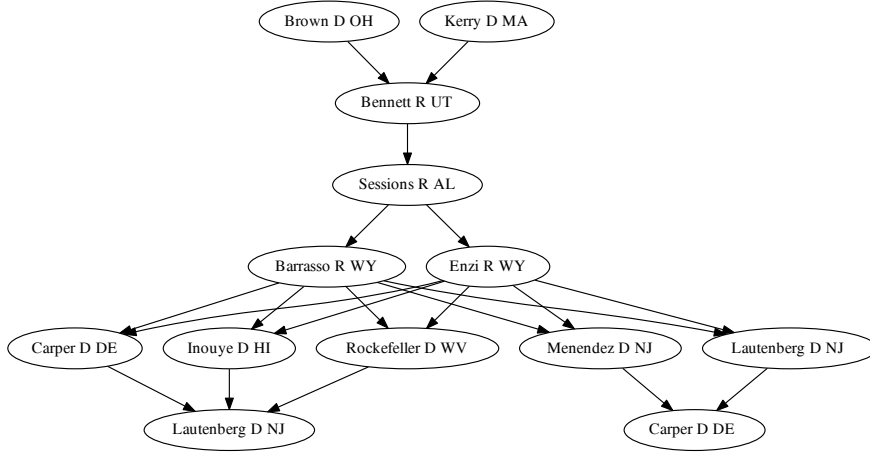


Figure 13: Most influential nodes in our setting. This directed acyclic graph (dag) illustrates all possible options for node selection that our approximation algorithm considers. A source node represents a node selected in the first iteration and a sink node represents a node selected in the last step. Any directed path from a source to a sink represents a sequence of nodes selected in successive iterations by our algorithm. All nodes in the same level and having the same parent, are tied in an iteration of the algorithm. Also note that the same node can appear in different paths of the dag at different levels.

²At the end, we also perform a post-processing step, where we try to remove one of the selected nodes to test if the remaining nodes are still most influential.

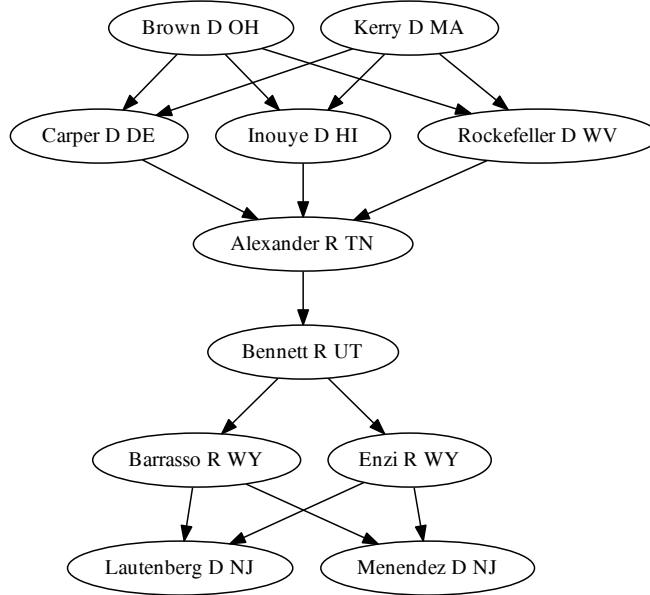


Figure 14: Most influential nodes in the diffusion setting. Each directed path from a source to a sink represents a sequence of nodes that

Although Figure 13 looks more complicated than Figure 14 (due to the appearance of the same node in different source-sink paths of the dag at different levels), comparing these we find that not only a set of six nodes are most influential in both cases but also most of the nodes are common between these two distinct frameworks. More remarkably, some of these common nodes are selected at the same iteration in both frameworks. One question that we can ask is does a set of most influential nodes in the LIG setting also remain most influential in the diffusion setting? We have exhaustively tested all possible sets of the most influential nodes (Figure 13) and settled the answer in the negative for each set. Interestingly, if we add the “Alexander R TN” node to *any* of the most influential sets in the LIG setting, the resulting set becomes most influential in the diffusion setting. The apparent similarity in results between the two models gives rise to an intriguing question asking us to connect these two mathematically and algorithmically different formulations, which is out of scope for this paper.

Filibuster

Beyond predicting stable behavior and identifying the most influential nodes in a network, our model can be used to study other interesting aspects of a networked population. One example is the filibuster phenomenon in the U.S. Congress, where a senator uses his or her right to hold floor for an indefinite time in an effort to delay the passing of a bill. It can be broken by the procedure of “cloture” which refers to gathering a majority of at least 60

votes among the current 100 senators. However, not every possible cloture scenario of 60 or more “yes” votes may be a *stable* outcome due to influence among the senators. The set of the ones that are indeed stable in the sense of PSNE will be called the *stable cloture set*.

An interesting general question that we can ask is whether there exists a small coalition of senators that can break filibusters. We can also think of preventing a filibuster from the democratic or the republican perspective (i.e., favoring the respective party). Let us formally define the problem.

Problem Formulation

Given the set \mathcal{S} of all stable outcomes (i.e., PSNE) and a subset of \mathcal{C} of these stable outcomes, find a minimal set T of players such that

$$T \in \arg \max_{V \subset \{1, \dots, n\}} \{|P_{\mathcal{S}}(V)| \text{ s.t. } P_{\mathcal{S}}(V) \subseteq \mathcal{C}\},$$

where $P_{\mathcal{S}}(V)$ is the set of PSNE-extensions of the nodes in V playing action 1, i.e.,

$$P_{\mathcal{S}}(V) = \{\mathbf{x} \text{ s.t. } \mathbf{x} \in \mathcal{S}, x_i = 1 \forall i \in V\}.$$

In words, \mathcal{C} is the stable cloture set, consisting of stable outcomes that can prevent a filibuster (i.e., every PSNE in \mathcal{C} contains at least 60 “yes” votes and thus, can induce a cloture). When we consider the notion of preventing filibuster that favors a specific party, \mathcal{C} is defined as consisting of exactly those PSNE that contain 60 or more “yes” votes (thereby representing cloture scenarios) and in addition, are supported (through “yes” votes) by the majority of the senators affiliated with that party. Other definitions are also possible as long as the stable cloture set \mathcal{C} is well-defined.

Now, we would like to select a *minimal* set of senators such that the set $P_{\mathcal{S}}(V)$ of the PSNE-extensions of these senators’ voting “yes” is contained in \mathcal{C} (i.e., their voting “yes” can only lead to a stable cloture scenario, thereby preventing filibuster). In addition, we would also like to achieve a maximum *stable-cloture cover*, that is, we wish to achieve the maximum possible set $P_{\mathcal{S}}(V)$ so that we are able to capture as many of the stable cloture scenarios as possible. In this formulation, we set up the objective to select a minimal, not minimum, set of senators in order to keep the formulation simple by avoiding bicriteria optimization (minimum set of senators vs. maximum stable-cloture cover). Further note that adding an extra senator to the set of selected senators can only reduce the stable-cloture cover due to additional constraints.

The above problem formulation guarantees a nonempty solution T if there exists some PSNE in \mathcal{C} that is not “dominated” by any PSNE in $\mathcal{S} \setminus \mathcal{C}$. Here, a PSNE \mathbf{x} dominates another PSNE \mathbf{y} if for every i , $y_i = 1 \implies x_i = 1$.

A Heuristic

We can modify the approximation algorithm for identifying the most influential nodes to design a heuristic for this problem in the following way. At each iteration, we select a node such that adding it to the set of already selected nodes minimizes the number of PSNE-extensions of the selected nodes playing 1 that are in $\mathcal{S} \setminus \mathcal{C}$. If there is a tie among several nodes in this step, then we can store these nodes in order to explore all solutions that this

heuristic can produce. We stop when the above number of PSNE-extensions within $\mathcal{S} \setminus \mathcal{C}$ goes to 0. We then perform a minimality test by excluding nodes from the selected set of nodes and testing whether the resulting set can be a solution. Note that although we can select the “best” solution (in terms of the coverage of \mathcal{C}) among the ones found due to ties, this heuristic does not guarantee an approximation of the maximum coverage of \mathcal{C} .

Experimental Results on the 110th Congress

For the 110th Congress, \mathcal{C} consists of 15,288 and 10,029 stable cloture scenarios (i.e., PSNE) with respect to the democratic and the republican parties, respectively. Overall, the total number of stable cloture scenarios is 15,595, and most of these are common in both democratic and republican cases. With respect to the democratic party, the best solutions found by the above heuristic are Senators {Brown (D, OH), Roberts (R, KS), and Graham (R, SC)} and {Kerry (D, MA), Roberts (R, KS), and Graham (R, SC)}, both of which cover 1,500 of the 15,288 stable cloture scenarios. The optimal solutions found by a brute-force procedure are Senators {Brown (D, OH), Craig (R, ID), and Dole (R, NC)} and {Kerry (D, MA), Craig (R, ID), and Dole (R, NC)}, both covering 1,728 stable cloture scenarios. With respect to republican party, the heuristic gives these two solutions as the best, each covering 40 of 10,029 stable cloture scenarios: Senators {Brown (D, OH), Bennett (R, UT), and Gregg (R, NH)} and Senators {Kerry (D, MA), Bennett (R, UT), and Gregg (R, NH)}. The optimal solution for this case is Senators {Bennett (R, UT), Conrad (D, ND), and Sessions (R, AL)}, which covers 138 stable cloture scenarios.

Application of Diffusion Models to this Problem

We can once again contrast our approach with that of diffusion to highlight two notable shortcomings of the latter. First, the notion of stable-cloture cover is not well-defined in the diffusion setting. The forward recursion mechanism central to diffusion models begins with a set of initial adopters (those senators selected to vote “yes” in our case) and propagates the effects of behavioral changes throughout the network until it reaches a steady state (i.e., no change occurs). However, this mechanism focuses on how the dynamics of behavioral changes evolves, not on the count of steady states that are consistent with a given set of players being among the adopters (not necessary early adopters), which is required for stable-cloture covers. In contrast, stable-cloture cover is well-defined in our approach.

Second and most important, even if we allow reversals of actions due to negative influence factors, forward recursion may produce an *unstable* outcome (i.e., not a PSNE). Although Granovetter’s original model precludes this by requiring the initial adopters to have a threshold of 0 (Granovetter, 1978), subsequent development allows forward recursion to start with a set of initial adopters whose thresholds are not necessarily 0 (Kleinberg, 2007). Next, we illustrate this point using our experimental results.

As justified above, in our experimental setting regarding diffusion models, we omit the notion of maximum stable-cloture cover and thereby forgo the measure of goodness of a solution. We only concentrate on finding a set of initial adopters that can drive the forward recursion process to *some* stable cloture scenario (i.e., a PSNE in \mathcal{C}). Our experimental procedure is outlined below.

For $k = 1, 2, \dots$, do the following. For all possible sets of k senators, start forward recursion with these k senators forced to play 1 all the time and other senators initially playing -1 (but are permitted to switch between 1 and -1 later on). When a steady state is reached, verify if there are at least 60 senators who are playing 1 in this state. If this is the case, then further verify if the k senators who are forced to play 1 are indeed playing their best response with respect to others' actions, which is the condition for the cloture scenario being stable. Stop iterating over k once stable cloture scenarios are found.

In our problem instances, which contain both positive and negative influence factors, it is very much possible that forward recursion oscillates indefinitely. However, it did not happen in our experiments. We tried all possible sets of $k \leq 3$ initial adopters, but failed to reach any cloture scenario (stable or unstable). We then tried all possible quadruplets of initial adopters. With respect to democratic party, 1,189 different quadruplets led the forward recursion process to a cloture scenario, but nearly half of these quadruplets (536 to be exact) led to unstable outcomes. Essentially, those unstable outcomes were due to some of the initial adopters not playing their best response in voting “yes”—all other nodes were indeed playing their best response (otherwise, the process would not terminate).

Therefore, beyond just emphasizing the stability of an outcome, our approach also captures certain phenomena that cannot be captured using the traditional approach.

7 Conclusion

In this paper, we studied influence and stable behavior from a new game-theoretic perspective. To that end, we introduced a rich class of games, named influence games, to capture the core strategic component of complex interactions in a network. We characterized the computational complexity of computing and counting PSNE in LIGs. We proposed practical, effective heuristics to compute PSNE in such games and demonstrated their effectiveness empirically. Besides predicting stable behavior, we gave a framework for computing the most influential nodes and its variants (e.g., identifying a small coalition of senators that can prevent filibuster). We also gave a provable approximation algorithm for the most influential nodes problem.

Although our models are inspired by earlier works by sociologists, at the heart of our whole approach is abstracting the complex dynamics of interactions by the solution concept of PSNE, which allowed us to deal with richer problem instances (e.g., the ones with negative influence factors) as well as to tread into new problem settings beyond identifying the most influential nodes. We conclude this paper by outlining several interesting lines of future work.

First, we leave several computational problems open. We have shown that counting the number of PSNE even in a star-type LIG is $\#P$ -complete, but does there exist an FPRAS for the counting problem? The computational complexity of indiscriminant LIGs, which we conjecture to be PLS-complete, is unresolved. Also, computing mixed-strategy Nash equilibria of LIGs, even for special types such as trees, remains an open question.

Second, we can apply our models to the general setting of “strategic interventions,” where we study the effects of changes in node thresholds, connectivity, or influence factors, usually without the possibility of having corresponding behavioral data. The following is an illustrative example of it in the context of the 111th U.S. Congress. After the death of

Ted Kennedy, who was a democratic senator from the state of Massachusetts, a republican named Scott Brown was elected in his place. Not only that it was Senator Brown’s first appointment in Senate, he was also the first republican from Massachusetts to be elected to Senate for a long time. Without any behavioral data at that time, we could perform interventions in our model under various assumptions of thresholds, connectivity, and influence factors regarding Senator Brown, with the general goal of predicting stable outcomes and investigating the effects of this intervention in various settings, such as the filibuster scenario or the setting of the most influential senators.

Another example of intervention, in the context of the Framingham heart study alluded in Section 1, is the following. Suppose that we would like to implement a policy of targeted interventions in order to reduce smoking by some margin. Using our model, we can modify the thresholds of the selected targets and predict how it could affect the overall level of smoking.

Besides interventions, we can also use our model to analyze past happenings, such as the role of the bipartisan “gang of six” senators in reaching an agreement during the U.S. debt ceiling crisis.³ We know that this bipartisan coalition did not succeed in its objective, but can we shed more light on it using the stable outcomes predicted by our model? Questions like these and many others shape the long-term goal of this research.

Notes and Acknowledgments

This article is a significantly enhanced version of an earlier conference paper (Irfan and Ortiz, 2011). Besides a more detailed exposition that places this work in the context of existing ones in sociology, economics, and computer science, the technical content has been extended in several ways. First, we have made the connection between linear influence games and polymatrix games. As a result, the computational complexity results have been shown to carry over to 2-action polymatrix games. Second, the complete proofs of the theoretical results have been given here.

In addition to enhancing the theoretical part, we have also extended the experimental part substantially. First and foremost, our approach to influence has been illustrated in a new setting originally suggested by Professor David C. Parkes (Harvard)—the U.S. Supreme Court rulings. Given the U.S. Supreme Court dataset, we first illustrate how we can learn a linear influence game to model the potential strategic interactions among the Supreme Court Justices using machine learning techniques (Honorio and Ortiz, 2012). We have then applied the schemes for PSNE computation and the identification of the most influential individuals in this new setting. Second, we have extended our empirical study of the U.S. congressional voting by contrasting our approach to that of diffusion. We have also applied our approach to a new problem of preventing filibuster by a coalition of senators, which highlights the broad range of scenarios in which our models can be used.

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³http://en.wikipedia.org/wiki/United_States_debt-ceiling_crisis

A Brief Review of Collective Behavior and Collective Action in Sociology

In sociology, the umbrella of *collective behavior* is very broad and encompasses an incredibly rich set of models explaining various aspects of a wide range of social phenomena such as revolutions, movements, riots, strikes, disaster, panic, diffusion of innovations (e.g., fashion, adopting contraceptives, electronic gadgets, or even religion) just to name a few. In fact, the richness of just one subfield of collective behavior, termed *micro-level theories of collective behavior*, led Montgomery to comment in his book (Montgomery, 1999, p. 67), “The variety of theories focusing on the micro level is confusing, but is an indication of the complexity and variations in the process by which movements emerge or perhaps fail to emerge...” Sociologists Marx and McAdam, in their concise introductory book on collective behavior (Marx and McAdam, 1994), contend that unlike many other fields of sociology, the field of collective behavior is not easy to define, partly because of the varied opinion of scholars, beginning from a very narrow perspective and ranging up to such a wide and all-encompassing perspective (e.g., Robert Park and Herbert Blumer’s) that there is virtually no need to have this as an individual field in sociology. Quoting from their book, “The field of collective behavior is like the elephant in Kipling’s fable of the blind persons and the elephant. Each person correctly identifies a separate part, but all fail to see the whole animal.”

Despite this, Marx and McAdam point out the traditional disposition to categorize collective behavior as a “residual field” in sociology. That is, the study of collective behavior consists of those elements of behavior (e.g., fads, fashion, crazes), organization (e.g., social movement), group (e.g., crowd), individual (e.g., psychological states such as panic), etc. that do not readily fit into well-established and commonly observed social structures. Similarly, collective behavior is defined in Goode’s textbook (Goode, 1992, p. 17) as the “relatively spontaneous, unstructured, extrainstitutional behavior of a fairly large number of individuals.”

Classical Treatment of Collective Behavior: Mass Hysteria

The classical treatment of collective behavior views individuals in a crowd as non-rational and transformed into being hysteric by the collective environment. The central tenet of the early work by Gustave Le Bon’s is that individuals in a crowd share a “mind of the crowd” and their psychological state and behavior in the collective setting is guided by the “psychological law of the mental unity of crowds” (Le Bon, 1897, p. 5). In Le Bon’s account, an individual in a crowd may retain some of the ordinary characteristics he shows in isolation, but the emphasis is on the extraordinary characteristics that emerge only in a crowd due to “a sentiment of invincible power,” contagion, and most importantly, due to the individuals being prone to taking suggestions as if they were hypnotic subjects. Examples of such extraordinary characteristics of a crowd are “impulsiveness,” “incapacity to reason,” and “the absence of judgment,” to name a few (Le Bon, 1897, p. 16). In sum, individuals in a crowd are depleted of their intellectual capacity and become uniform in their psychological state. This leads the crowd to an identical direction of collective behavior that may be heroic or criminal, depending on the type of “hypnotic suggestion”

alluded above (although Le Bon gives examples of heroic crowds (Le Bon, 1897, p. 14), for the most part, he rather portrays crowds with a negative connotation).

Le Bon’s work influenced subsequent developments for around half a century. His proposition of transformation of individuals in a crowd was upheld by Park and Burgess, who put forward the concept of *circular reaction* (Park and Burgess, 1921), which was later refined by Herbert Blumer (Blumer, 1939). Circular reaction refers to a reciprocal process of social interaction that explains *how* crowd members become uniform in their behavior, something that Le Bon could not really explain. In this process, an individual’s behavior stimulates another individual to behave alike, and when the latter individual does so, it reinforces the stimulation that the former individual acted upon. In circular reaction, individuals do not act rationally or intellectually. That is, they do not reason about the action of others; they only align themselves with the behavior of others. This is different from *interpretative interaction*, another mechanism that Blumer defined to explain routine group behavior (e.g., a group of individuals shopping in a mall) as opposed to collective behavior (e.g., social movement). In interpretative interaction, individuals react (perhaps differently) to their *interpretation* of others’ action, not the action itself. Therefore, individuals can be treated as rational beings in interpretative interaction. (For clarity of presentation, we differentiate between routine group behavior and collective behavior. To the contrary, Blumer, as well as Park in his earlier work, viewed collective behavior as encompassing a wide range of social phenomena, including routine group behavior. Within the continuum of collective behavior in Blumer’s view, the presence or the absence of rationality of individuals earmarks two distinct mechanisms named interpretative interaction and circular reaction, respectively.)

In Blumer’s account, a crowd goes through several well-defined stages before a collective behavior finally emerges. The three underlying mechanisms that facilitate transitions among these stages are circular reaction, collective excitement, and social contagion, where one can roughly think collective excitement as a more intense form of circular reaction and social contagion as even more intense (McPhail, 1991, p. 11).⁴

Emergent Norm Thoery

Although Blumer’s account of collective behavior received wide-spread acceptance even beyond academia (McPhail, 1991, p. 9), many of the underlying assumptions in it, as well as in the general mass hysteria theory, were deemed unrealistic by others. Arguably, a collective behavior is participated by individuals with *different* objectives in mind, and *changes* in their individual behavior can be observed throughout the process of a collective behavior. The reader is referred to (Miller, 2000, p. 26–27) for a beautiful example in the context of the 1967 anti-war demonstration in Washington, DC, which was participated by nearly 250,000 people. While many of the participants might have been there to genuinely voice their opinion against the war, some might have been looking for “excitement, drug, or sex.” Yet again, individuals playing different roles, such as protest leaders, street vendors, and the police, behaved differently. Therefore, the assumption of complete uniformity

⁴The mechanism of “social contagion” as defined by Blumer or the “social contagion theory” (Locher, 2001, p. 11) in general is not to be confused with the term “social contagion” that computer scientists use (Ugander et al., 2012). Although both have their roots in epidemics, in the former case, individuals are transformed into being more suggestible that facilitates “rapid, unwitting, and non-rational dissemination” of behavior (Blumer, 1939), whereas in the latter case, individuals act rationally.

behavior in the classical mass hysteria treatment is very much a stretch. Furthermore, the assumption of hysteric crowd in the classical approach has also been called into question. One notable critique of the mass hysteria theory comes from Ralph Turner and Lewis Killian (Turner and Killian, 1957). They view individuals in a crowd as behaving under normative constraints and showing “differential expression.” However, when the crowd is faced with an extraordinary situation that is not adequately guided by the established norms of the society, a new norm emerges. They call this the *emergent norm* and contend that it is the emergent norm that gives the “illusion of unanimity.”

Collective Action

The goal-oriented nature of collective behavior was further highlighted by sociologists studying social movements during the 1970s and 80s. In order to distinguish their approach from the traditional approach to collective behavior dominated by the assumption of irrational and aimless nature of crowds, they used the term *collective action* signifying “people acting together in pursuit of common interests” (Tilly, 1978). Strikingly, based on a series of systematic observations, Clark McPhail’s contends that the goal-oriented nature of crowds is not limited to social movements and revolutions alone, but is a feature of various other types of crowds. In his book *the Myth of the Madding Crowd*, he uses two decades of empirical observations pertaining to a multitude of crowd settings to formulate a theory of collective behavior that is recognized as a significant paradigm shift (McPhail, 1991, ch. 5, 6). To distance himself from the term “crowds,” which has already gained several meanings depending on whose theory is being considered, he gives his formulation in the setting of “gatherings.” But first, he places a justifiably strong emphasis on the definition of collective behavior. His “working definition of collective behavior” can be stated briefly as the study of “two or more persons engaged in one or more behaviors (e.g., locomotion, ...) judged common or concerted on one or more dimensions (e.g., direction, velocity, ...)”⁵ (McPhail, 1991, p. 159).

The broad nature of McPhail’s definition, although based on extensive empirical evidence, was not readily accepted as a definition of collective behavior. Even modern textbooks on collective behavior try to conserve the classical appeal of collective behavior. For example, in reference to McPhail’s definition, Goode writes in his textbook, “In the view of most observers, myself included, many gatherings are *not* sites of collective behavior (most casual and conventional crowds, for example), and much collective behavior does not take place in gatherings of any size (the behavior of most masses and publics, for example)” (Goode, 1992, p. 17). Ironically, this is the very viewpoint that McPhail seeks to portray as a myth. Perhaps to further distance himself from the traditional viewpoint, McPhail later began to use the term *collective action* instead of collective behavior (for example, in a recent encyclopedia article, McPhail refers to the above mentioned definition as that of collective action (McPhail, 2007)). According to David Miller, the modern view on the distinction between collective action and collective behavior is beyond simply terminological. Collective action is given the status of a “new” theory in sociology, while collective

⁵Note that it is the “behavior,” not a specific action, that needs to be common or concerted. For example, when a group of people are chatting together, their behavior is concerted, even though they are not speaking identical words.

behavior is marked as “old,” but not unimportant. (Miller, 2000, p. 14–15). Miller also points out that sociology textbooks are likely to talk about collective behavior only whereas recent journal articles on collective action.

McPhail’s approach to collective action is known as *social behavioral interactionist (SBI)* approach. As much as it agrees with the emergent norm theory in terms of the diversity of individual objectives in a collective setting, it does not agree with the concept of an emergent norm suppressing this diversity. The SBI approach studies gatherings in three phases of its life cycle: the assembling process, collective action within the assembled gathering, and the dispersal process (McPhail, 1991, p. 153). Although each of these three phases is rich and interdependent, the goal is to manage the complexity of collective action as a whole by focusing on the recognizable parts of it. Interestingly, the underlying mechanism to explain collective action is drawn from the *perceptual control theory* (McPhail, 1991, Ch. 6). McPhail adapts this theory to formulate his *sociocybernetics theory* of collective action. In brief, an individual receives sensory inputs, compares the input signal to its desired signal,⁶ and adjusts its behavior in response to the discrepancy. Behavior of individuals affects the “environment,” which in turn affects the input signal, thereby completing a loop. An important aspect of this theory is that various external factors (or “disturbances”) may drive an individual to make different behavioral adjustments at different points in time even if the discrepancy between the input signal and the desired signal remains the same.

Additional Notes on Schelling’s Models

Schelling’s models assume that individuals behave in a discriminatory way. For example, individuals are aware, consciously or unconsciously, about the types of other individuals in their neighborhood and behave (i.e., stay in the neighborhood or leave) according to their preference. This is different from organized processes (e.g., separation of on-campus residence between graduate and undergraduate students due to a university’s housing policy) or economic reasons (e.g., segregation between the poor and the rich in many contexts) (Schelling, 1971, 1978). An example of a segregation due to individual choice, or “individually motivated segregation” as Schelling puts it (Schelling, 1971, p. 145), is the residential segregation by color in the U.S. In fact, Schelling’s models and their analyses expressly focus on this case. Yet, Schelling’s theory can be applied to many other scenarios as well, since it explains, at an abstract level, how *collective* outcomes are shaped from *individual* choice. It should be mentioned here that connecting individual actions to collective outcomes is a mainstream theme of research in collective action.

Schelling introduces two basic models to study the dynamics of segregation among individuals of two types (Schelling, 1971). The first model is named the *spatial proximity model* where individuals are initially positioned in a spatial configuration (such as a line or a stylized two-dimensional area) and individuals of the same type share a common “level of tolerance,” which quantifies the upper limit on the percentage of an individual’s opposite type in his local neighborhood that he can put up with. Here, an individual’s local neighborhood is defined with reference to the individual’s position in the specified spatial configuration. The dynamics of segregation is studied in this model using a rule of movement

⁶In contrast to engineering control systems theory, the desired signal is not external, but set by individuals themselves (which is also highlighted by the term *cybernetics*).

for the “unhappy” individuals. For example, an individual whose level of tolerance has been exceeded, moves to the closest location where the tolerance constraint can be satisfied. For equal number of individuals of each type and a fixed local neighborhood size, Schelling first studies how clusters evolve from the initial configuration of a random placement of individuals on a straight line. He then generalizes the experimentation by varying various model parameters such as neighborhood size, level of tolerance, ratio of individuals of the two types, etc. Notable findings are that decreasing the local neighborhood size leads to a decrease in the average cluster size and that for unequal number of individuals of the two types, decreasing the relative size of the minority leads to an increase in the average minority cluster size.

Schelling extends this experimentation to a different setting of a two-dimensional checkerboard. The individuals are randomly distributed on the squares of the checkerboard, leaving some of the squares unoccupied. An individual’s local neighborhood is defined by the squares around it and an unhappy individual moves to the “closest” unoccupied square (leaving its original square unoccupied) that can satisfy its tolerance constraint. In addition to studying clustering properties by varying various model parameters, two new classes of individual preferences have been studied—congregationist preferences and integrationist preferences. In a congregationist preference, an individual only wants to have at least a certain percentage of neighbors of its own type and does not care about the presence of individuals of the opposite type in its neighborhood. Experimentation shows that even when each individual is happy being a minority in its neighborhood (e.g., having three neighbors of its own type out of eight), the dynamics of segregation leads to a configuration as if the individuals wished to be majority in their neighborhoods. In an integrationist preference, individuals have both an upper and a lower limit on the level of tolerance. Dynamics is much more complex in this case and leads to clusters of unoccupied squares.

Schelling’s second model, named the *bounded-neighborhood model*, is concerned with one global neighborhood. An individual enters it if it satisfies its level of tolerance constraint and leaves it otherwise. The level of tolerance is no longer fixed for each type and the distribution of tolerances among individuals of each type is given. The emphasis on the stability of equilibria when the distribution of tolerances and the population ratio of the two types are varied. For example, under a certain linear distribution of tolerances and a population ratio of 2 : 1, there exist only two stable equilibria, each consisting of individuals of one type only, whereas a mixture of individuals of both types can arise as a stable equilibrium under a different setting. This model has been adapted for the study of the *tipping* phenomenon with one notable constraint, that is, the capacity of the neighborhood is fixed. An example of a tipping phenomenon is when a neighborhood consisting of only one type of individuals is later inhabited by some individuals of the opposite type and as a result, the entire population of the original type evacuates the neighborhood. An important finding is that in the cases studied, the modal level of tolerance does not correspond to a tipping point.

B Experimental Results in Tabular Form

Table 2 shows experimental data of PSNE computation on uniform random directed graphs. Of particular interest is the result that the number of PSNE usually increases when the flip

Table 2: PSNE computation on uniform random directed graphs. Offsets of 95% confidence intervals are shown in parenthesis.

p	# of equilibria Avg (95% CI)		# of node visits/equilibrium Avg (95% CI)		Avg CPU time (sec) for computing all equilibria
0.00	2.18	(0.16)	35379.73	(3349.77)	1.81
0.125	3.72	(0.50)	22756.15	(2673.56)	1.57
0.25	13.00	(1.92)	9796.30	(1748.76)	1.9
0.375	19.42	(2.88)	7380.97	(1870.11)	1.95
0.50	14.40	(2.17)	9826.61	(1696.52)	2.04
0.625	19.78	(3.40)	8167.60	(1450.48)	2.07
0.75	28.76	(4.34)	6335.18	(1963.11)	2.21
0.875	67.14	(9.52)	4064.06	(1539.33)	3.27
1.00	194.96	(28.47)	1879.45	(235.90)	5.23

probability p is increased, i.e., when the number of arcs with negative influence factors is increased.

Table 3 illustrates the experimental result that the logarithmic-factor approximation algorithm for identifying the most influential individuals performs very well in practice.

Finally, Table 4 shows the influence factors and thresholds of the LIG among the U.S. Supreme Court Justices, which are learned using the U.S. Supreme Court dataset.

Table 3: Computation of the most influential nodes: Comparison between the solution given by the ApproximateUniqueHyperedge algorithm and the optimal one in random directed graphs. Offsets of the 95% confidence intervals are shown in parenthesis.

p	Approx soln size		Opt soln size		Approx = Opt	Approx \leq Opt+1	Approx \leq Opt+2
	Avg (95% CI)		Avg (95% CI)		% of times	% of times	% of times
0.00	1.06	(0.05)	1.06	(0.05)	100	100	100
0.125	1.50	(0.11)	1.48	(0.10)	98	100	100
0.25	2.57	(0.18)	2.14	(0.11)	62	96	99
0.375	2.75	(0.18)	2.35	(0.13)	62	98	100
0.50	2.47	(0.16)	2.09	(0.10)	66	96	100
0.625	2.82	(0.18)	2.31	(0.13)	52	97	100
0.75	3.02	(0.19)	2.48	(0.14)	51	95	100
0.875	3.83	(0.20)	3.04	(0.15)	36	86	99
1.00	4.72	(0.23)	3.95	(0.18)	37	87	99

	Scalia	Thomas	Rehnquist	O'Connor	Kennedy	Breyer	Souter	Ginsburg	Stevens
Scalia	0.0120	0.4282	0.0317	0.0717	0.0721	0.0772	-0.0321	0.1362	-0.1388
Thomas	0.2930	-0.1020	0.1245	-0.0010	0.0183	-0.1497	0.0839	-0.1311	0.0965
Rehnquist	0.0671	0.1762	-0.0834	0.0973	0.1254	0.0921	-0.0861	0.1336	-0.1388
O'Connor	0.0580	0.1073	0.1045	-0.2522	-0.0537	0.2313	0.0325	-0.1245	-0.0359
Kennedy	0.0666	0.1236	0.1863	-0.0255	-0.2634	0.0548	-0.1115	-0.0149	0.1532
Breyer	0.1009	-0.2191	0.0570	0.2208	-0.0061	-0.0209	0.0627	0.1102	0.2023
Souter	0.0368	0.1192	-0.0476	0.0762	-0.0338	0.1429	-0.0619	0.2783	0.2034
Ginsburg	0.0779	-0.1000	0.1613	-0.0962	-0.0089	0.1199	0.1999	-0.0381	0.1978
Stevens	-0.1379	0.1088	-0.1721	-0.0568	0.1053	0.1374	0.0932	0.1274	-0.0611

Table 4: LIG learned from data. Each non-diagonal element in each row represents the influence factor the row player receives from the column players (for example, Justice Scalia's influence factor on Justice Thomas is 0.2930 and that in the opposite direction is 0.4282). The diagonal elements represent thresholds of the corresponding row players.

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